A Scott Conjecture for Hyperbolic Groups

Jeremy Shor
Columbia University
shor@math.columbia.edu

For a given automorphism ϕ of a group G define $Fix(\phi) = \{g \in G : \phi(g) = g\}$. A subgroup H of G is a *fixed subgroup* if there is an automorphism ϕ of G with $H = Fix(\phi)$. Our main result is the following:

Theorem: *A torsion free hyperbolic group contains, up to isomorphism, only finitely many fixed subgroups.*

The Scott Conjecture, which was proven by M. Bestvina and M. Handel [BH], states that the rank of fixed subgroups in a free group of rank n, F_n, are at most n. In particular, F_n contains, up to isomorphism, only finitely many fixed subgroups. Our result may be viewed as an extension of this last statement to the class of torsion free hyperbolic groups. Fixed subgroups of hyperbolic groups have been investigated by Paulin [P1] and Neumann [N]. The first shows that fixed subgroups are always finitely generated and the second shows they are always rational, which implies hyperbolicity.

In [S], Sela reproves the Scott Conjecture. It is his proof on which ours is based. Using work of Paulin [P2], Sela is able to isolate the subgroup where an automorphisms is periodic. To be exact, he shows that an automorphism of a free group is periodic on a vertex group in a graph of groups decomposition with trivial or infinite cyclic edge groups. He also establishes this result for freely indecomposable torsion free hyperbolic groups, and this is our starting point.

I would like to thank my thesis advisor, Mark Feighn for his support and many helpful conversations throughout the course of this work. I would also like to thank Mladen Bestvina for many helpful suggestions.

SECTION 1

In this section we collect some definitions and recall some results that will be needed in the course of our proof. We assume the reader is familiar with the notation and terminology of the Bass-Serre theory of group actions on trees, as well as the JSJ decomposition of a torsion free, freely indecomposable hyperbolic group.

Definition 1.1: A group G is *freely indecomposable* if $G = A * B$ implies A or B is trivial.

Definition 1.2: Recall that JSJ decomposition is a graph of groups presentation of a group. Call a tree T associated to a JSJ decomposition of G a JSJ *tree* for G.

Definition 1.3: Suppose G acts by isometries on a simplicial tree T. Let $Aut(G)$ denote the automorphism group of G. Denote by $Aut(G, T)$ the subgroup of $Aut(G)$ consisting of all automorphisms ϕ such that whenever $H \subset G$ stabilizes a point of T so does $\phi(H)$.

If we have a G tree T with infinite cyclic edge stabilizers and it was not the case that $Aut(G) = Aut(G, T)$ we could then let G act on T via some automorphism not in $Aut(G, T)$. We then would get a new splitting of G over infinite cyclic groups. This can't be the case, however, if T is a JSJ tree for G. Hence we have:

Proposition 1.4: *If G is a freely indecomposable group and T a JSJ tree for G then $Aut(G) = Aut(G, T)$.*

We will also need a result of [Gu] which characterizes the Bass-Serre splittings of a hyperbolic group G over infinite cyclic edge groups.

Definition 1.5: Let G be a freely indecomposable group. Call $H \subset G$ a *Z-factor* if H is the stabilizer of a point in some G tree for which has infinite cyclic edge stabilizers.

Definition 1.6: Suppose a group G has a graph of groups decomposition with associated tree T. Choose an edge in the underlying graph and collapse all of its lifts in T to a point. The result is a new G tree T'. This produce a new graph of groups decomposition for G. If we perform a sequence of these moves on T we say we are *blowing down* T. Call the reverse procedure *blowing up* T. In this paper *blowing up* will always consist of new infinite cyclic splittings realized by simple closed curves on the surfaces underlying quadratically hanging subgroups.

Definition 1.7: Suppose G acts on a Bass-Serre tree T. If H is a finitely generated subgroup of G denote by T_H the minimal subtree for H in T. Also, if $e \subset T$ is a subset of T denote by $Stab(e)$ the subgroup of G which fixes e point wise.

Theorem 1.8 [Gu]: *Suppose H is a $\mathbb{Z}$-factor of G with S its associated G tree. There is a JSJ tree T for G which can be blown up and then blown down into S.*

Definition 1.10: Let $Per(\phi) = \{g \in G : \phi^i(g) = g \text{ for some integer } i\}$, the subgroup of G where ϕ is periodic.

The next fact we will need is a restatement of a theorem of Sela. His Theorem 3.2 [S] along with the observations of how $Per(\phi)$ must act on his limit trees (in Theorem 4.1 of the same paper) give us the following fact:

Theorem 1.11 [S]: *Suppose G is a torsion free, freely indecomposable hyperbolic group. Either $Fix(\phi)$ is trivial, infinite cyclic or $Per(\phi)$ is a $\mathbb{Z}$-factor of G.*

Definition 1.12: Recall $Out(G) = Aut(G)/Inn(G)$ where $Inn(G)$ are the inner automorphisms of G. Also if $H \subset G$ denote by $Aut_G(H)$ the automorphisms of H which extend to automorphisms of G.

The following fact is crucial for us and is indeed the reason we are 'better off' once we have 'reduced' our question to a question about the JSJ vertex groups.

Proposition 1.13 [P] [S]: Let G be a freely indecomposable hyperbolic group and V a subgroup of G which stabilizes some point in a JSJ tree of G. Also suppose V is not a quadratically hanging subgroup. It is then the case that $Aut_G(V)$ has finite image in $Out(V)$.

SECTION 2

We prove our main theorem assuming three lemmas. The proofs of the three lemmas are contained in the remaining sections.

Theorem: *A torsion free hyperbolic group G contains, up to isomorphism, only finitely many fixed subgroups.*

Lemma 1: *It is enough to prove our main theorem in the case that G is freely indecomposable.*

Lemma 2: *Suppose G is a freely indecomposable torsion free hyperbolic group and $\phi \in Aut(G)$. Either $Fix(\phi)$ is trivial, infinite cyclic or $Fix(\phi)$ receives a graph of groups presentation $\mathcal{G}$ with the following properties:*

- *(i) Edge groups are trivial or infinite cyclic.*
- *(ii) There is a bound k, depending only on G so that the underlying graph of $\mathcal{G}$ has at most k edges.*
- *(iii) Vertex groups are trivial, infinite cyclic or the fixed subgroup of ϕ restricted to a vertex stabilizer in some JSJ tree for G.*

Definition 2.1: Let L_1 and L_2 be finite collections of conjugacy classes in groups H_1 and H_2. Say (H_1, L_1) is equivalent to (H_2, L_2), written $(H_1, L_1) \sim (H_2, L_2)$, if there is an isomorphism from H_1 to H_2 which maps the conjugacy classes in L_1 bijectively to L_2. This is seen to be an equivalence relation.

Definition 2.2: Suppose G is a freely indecomposable torsion free hyperbolic group. Let $\mathcal{U}$ be the set of all graph of groups presentations , over all $\phi \in Aut(G)$, that $Fix(\phi)$ recieves (as in lemma 2), for which $Fix(\phi)$ is not trivial or infinite cyclic. Let $\mathcal{V}$ be the set of pairs (H, L), where H is a vertex stabilizer in a tree associated to some graph of groups presentation in $\mathcal{U}$, and L is the conjugacy classes (in H) of all edge groups incident to H.

Lemma 3: *Suppose G is a torsion free, freely indecomposable hyperbolic group. It is the case that $|\mathcal{V}/\sim|$ is finite.*

Proof of Theorem: By Lemma 1 we may assume G to be freely indecomposable. By Lemma 2, for each $\phi \in Aut(G)$, $Fix(\phi)$ receives a graph of groups presentation with at most k edges. This implies that, over all $\phi \in Aut(G)$ with fix subgroup not trivial or infinite cyclic, there are only finitely many different underlying graphs in these graph of group presentations.

It is enough to show our claim over each possible underlying graph. By Lemma 2 *(i)* and Lemma 3 we know there are only 2 possible edge groups and finitely many different possible vertex groups. Hence given an underlying graph there are only a finite number of ways of assigning edge and vertex groups. We are done if we show that only finitely many fundamental groups occur from a specific way of adorning our graph, where the freedom comes from different possible edge group monomorphisms. We now show that with Lemma 3 this is indeed the case.

Choose a vertex v along with its incident edges e_1, e_2, ... e_n in our graph K. Suppose H is associated to v and an infinite cyclic group is associated to each of e_1, e_2, ... e_i. By Lemma 3 there are only finitely many ways, up to isomorphism of H and choice of conjugacy class, of picking the images of the bonding maps for the edge groups. Further this is the case for each vertex of K.

Hence we may assume it is enough to show our claim in the case that we have a) one underlying graph, b) one way of adorning it with edge and vertex groups, and c) that at each vertex there is one way (up to isomorphism of the vertex group and conjugacy class for the edge group in the vertex group) of choosing our edge group monomorphisms for the edges which have the infinite cyclic group associated to them. Given two graphs of groups decompositions which satisfy a)-c) we can easily construct an isomorphism between there respective fundamental groups. We do this by extending the obvious graph bijection to isomorphisms of the associated edge and vertex groups. ♠

SECTION 3

In this section we prove the following lemma:

Lemma 1: *It is enough to prove our main theorem in the case that G is freely indecomposable.*

In [CT], Collins and Turner prove a free decomposition version of the Scott Conjecture. It is their result which will allow us to reduce to the freely indecomposable case.

Let G be a group with $G = F_r * (\mathop{*}\limits_{i=1}^{t} G_i)$, where each G_i is a non free, freely indecomposable group and F_r is the free group on r elements. The Kuros subgroup theorem states that if H is a subgroup of G then $H = F_s * (\mathop{*}\limits_{i=1}^{u} H_i)$, where the H_i are the intersections of H with conjugates of the factor groups G_i, F_s is free with rank s, and F_s does not intersect any conjugate of any G_i. By [BL] the following definition is well defined.

Definition 3.1: Suppose $H \subset G$ and $G = F_r * (\mathop{*}\limits_{i=1}^{t} G_i)$ where each G_i is a non free, freely indecomposable group and F_r is the free group on r elements. Also suppose $H = F_s * (\mathop{*}\limits_{i=1}^{u} H_i)$, where the H_i are the intersections of H with conjugates of the factor groups G_i, F_s is free with rank s, and F_s does not intersect any conjugate of any G_i. Define $s+u$ to be the *Kuros subgroup rank* of H in G denoted by $KRk(G, H)$. We also say $KRk(G, G) = r + t$

One of the main results of [CT] is the following:

Theorem 3.2 [CT] For any $\phi \in Aut(G)$

$$KRk(G, Fix(\phi)) \leq KRk(G, G).$$

Proof of Lemma 1: Suppose G is a torsion free hyperbolic group. Also suppose $G = F_r * (\mathop{*}\limits_{i=1}^{t} G_i)$ where each G_i is a non free, freely indecomposable group and F_r is the free group on r. By Kuros $Fix(\phi) = F_s * (\mathop{*}\limits_{i=1}^{u} S_i)$ where where the S_i are the intersections of $Fix(\phi)$ with conjugates of the factor groups G_i, F_s is free with rank s, and F_s does not intersect any conjugate of any G_i. If $Fix(\phi)$ intersects some freely indecomposable factor of G then ϕ must restrict to an automorphism of that factor. Hence, assuming our main theorem is true for freely indecomposable torsion free hyperbolic groups, there are only finitely many different possibilities for the S_i.

By Theorem 3.1 s+u $\leq$ r+t and there are only finitely many possibilities for $Fix(\phi)$.
♠

SECTION 4

In this chapter we prove the following lemma:

Lemma 2: *Suppose G is a freely indecomposable torsion free hyperbolic group and $\phi \in Aut(G)$. Either $Fix(\phi)$ is trivial, infinite cyclic or $Fix(\phi)$ receives a graph of groups presentation $\mathcal{G}$ with the following properties:*

(i) Edge groups are trivial or infinite cyclic.

(ii) There is a bound k, depending only on G so that the underlying graph of $\mathcal{G}$ has at most k edges.

(iii) Vertex groups are trivial, infinite cyclic or the fixed subgroup of ϕ restricted to a vertex stabilizer in some JSJ tree for G.

We assume throughout this chapter that G is a torsion free, freely indecomposable hyperbolic group whose JSJ decomposition contains n edges. By Theorem 1.11 for every $\phi \subset Aut(G)$ $Fix(\phi)$ is trivial, infinite cyclic or $Per(\phi)$ is a $\mathbb{Z} - factor$ of G. Suppose $Per(\phi)$ is a $\mathbb{Z} - factor$ of G. By Theorem 1.8 there is a JSJ tree T for G which can be blown up and then down into a tree realizing $Per(\phi)$ as a $\mathbb{Z} - factor$ of G. We will show that the action of $Fix(\phi)$ on $T_{Fix(\phi)} = (T_{Per(\phi)})_{Fix(\phi)}$ induces a graph of groups presentation of $Fix(\phi)$ which satisfies the conclusions of Lemma 2.

First some facts about the dynamics of the edge stabilizers on the Bass-Serre tree T.

Definition 4.1: A subgroup $H \subset G$ is *malnormal* if $gHg^{-1} \cap H \neq 1$ implies $g \in H$.

Remark 4.2: It is shown in [G] that elements in a torsion free hyperbolic group have infinite cyclic normalizers which are malnormal. This implies that if two elements (one of which is non trivial) commute or share a common power then the subgroup they generate is infinite cyclic.

Lemma 4.3: *Let $< g >$ be an infinite cyclic subgroup in a torsion free hyperbolic group and let $\phi \in Aut(G)$. If $\phi(g^n) \in < g >$ then $\phi(g^n) = g^n$ and $\phi(g) = g$, or $\phi(g^n) = g^{-n}$ and $\phi(g) = g^{-1}$.*

Proof: If $\phi(g^n) \in < g >$ then $\phi(g)\phi(g^n)\phi(g)^{-1} \in < g >$. This implies $\phi(g) < g > \phi(g)^{-1}$ non trivially intersects $< g >$. Malnormality implies that $< \phi(g), g >$ is infinite cyclic. If $\phi(g)$ then is not g or g^{-1} we are able to construct, an ascending chain of infinite cyclic subgroups. Since this cannot occur in a hyperbolic group we are done. ♠

Lemma 4.4: *Let e_1, e_2 and e_3 be three consecutive edges in T with edge stabilizers $< g_1 >, < g_2 >$ and $< g_3 >$. One of the following possibilities holds:*

(i) $< g_1 >$ *and* $< g_2 >$ *intersect trivially, or,*

(ii) There is a $g \in G$ with both $< g_1 >$ and $< g_2 >$ contained in $< g >$. If $< g_2 >$ is properly contained in $< g >$ then $< g_3 >$ is contained in $< g_2 >$, or $< g_3 >$ and $< g_2 >$ intersect trivially.

Proof: Suppose $< g_1 >$ and $< g_2 >$ intersect non trivially. If $g_1^n \in < g_2 >$ then ${g_1}^n = {g_2}^m$, and by the above remark, $< g_1, g_2 >$ is infinite cyclic, and hence there is a $g \in G$ with $< g_1 >$ and $< g_2 >$ contained in $< g >$. If $< g_2 >$ is properly contained $< g >$ then g does not stabilize e_2. Now suppose $< g >$ is not $< g_2 >$, $< g_2 >$ intersects $< g_3 >$ non trivially and $< g_3 >$ is not contained in $< g_2 >$. By the same argument there is an $h \in G$ with $< g_2 >$ and $< g_3 >$ contained in $< h >$. Now h does not stabilize e_2 since $< h >$ properly contains $< g_2 >$. Now h and g normalize $< g_2 >$ and by our above remark $< g, h >$ must be infinite cyclic. This is a contradiction because if an infinite cyclic group acts on a tree with some element acting elliptically (g_2) then the whole infinite cyclic group fixes one point. But the infinte cyclic group generated by h and g conatains g_1 and g_3, which by assumptions stabilize different regions of T. ♠

Recall that by Proposition 1.4 $Aut(G) = Aut(G, T)$. This allows us to create, from an automorphism ϕ of G, a continuous map ϕ_* of T to itself. The details of this construction along with some observations follow.

Definition 4.7 : Suppose v_1 and v_2 are distinct vertices of T. Denote by $\overline{v_1 v_2}$ the unique geodesic in S bounded by v_1 and v_2.

Construction 4.8: Let T be a JSJ tree for G and ϕ an automorphism of G. Let $F \subset T$ be a lift of the quotient graph T/G. For a vertex $v \in F$ suppose V is the subgroup of G which stabilizes it. Since $Aut(G) = Aut(G, T)$, $\phi(V)$ stabilizes some vertex $w \in T$. Define $\phi_*(v) = w$. Define ϕ_* for each vertex in F in this fashion and then extend over the rest of the vertices of T G-equivariantly. We conclude by defining ϕ_* on the edges of T. Suppose v_1 and v_2 are adjacent edges in T. Expand ϕ_* to $\overline{v_1 v_2}$ by linearly stretching $\overline{v_1 v_2}$ over $\overline{\phi(v_1)\phi(v_2)}$.

Definition 4.9: We have that ϕ_* is a continuous map of T to itself which maps vertices to vertices. If $\overline{v_1 v_2} \subset \overline{\phi_*(v_1)\phi_*(v_2)}$ with orientations agreeing say $\overline{v_1 v_2}$ is an *expanding arc*. We note that $\overline{v_1 v_2}$ may equal $\overline{\phi(v_1)\phi(v_2)}$ and still be an expanding arc.

Lemma 4.10: *Suppose each edge in T is declared to have length one. If $\overline{v_1v_2}$ is an expanding arc then $\overline{v_1v_2}$ contains a length one expanding arc.*

Proof: We induct on the length of $\overline{v_1v_2}$. If the length of $\overline{v_1v_2}$ is one then we are done. So assume every expanding arc with length less than n contains a length one expanding arc. Suppose $\overline{v_1v_2}$ has length n. Choose a vertex $w \in \overline{v_1v_2}$ not equal to v_1 or v_2. Let v be the point on $\overline{v_1v_2}$ which is closest to $\phi_*(w)$. Without loss of generality we can assume that v either, is v_1 or w, or separates w from v_1 on $\overline{v_1v_2}$. It is then the case that $\overline{w, v_2}$ is an expanding arc with with length less than n, and by our induction hypothesis we are done. ♠

Lemma 4.11: *If $\overline{v_1v_2}$ is an expanding arc then*

$$\phi(Stab(\overline{v_1v_2})) \subset Stab(\overline{v_1v_2}).$$

Proof: If $\overline{v_1v_2} \subset \overline{\phi_*(v_1)\phi_*(v_2)}$ then $Stab(\overline{\phi_*(v_1)\phi_*(v_2)}) \subset Stab(\overline{v_1v_2})$. We know that $Stab(\overline{\phi_*(v_1)\phi_*(v_2)}) = Stab(\phi_*(v_1)) \cap Stab(\phi_*(v_2))$ which contains $\phi(Stab(v_1)) \cap \phi(Stab(v_2))$ which is equal to $\phi(Stab(\overline{v_1v_2}))$. ♠

Remark 4.12: It is shown in [CM] that there are four possible types of non trivial group actions on a tree. They are linear, dihedral, parabolic or hyperbolic. Dihedral actions imply the existance of torsion elements, while parabolic actions on a tree with infinite cyclic edge stabilizers implies the group is solvable. Recall that in a hyperbolic group a solvable subgroup is virtually infinite cyclic [G]. Hence subgroups of a one ended torsion free hyperbolic group admit only linear or hyperbolic actions on JSJ trees.

Lemma 4.13: *If $\phi \in Aut(G)$ then one of the following holds:*

(i) The action of $Fix(\phi)$ on T is trivial or linear, or,

(ii) The action of $Fix(\phi)$ on $T_{Fix(\phi)}$ is hyperbolic, and, there is a length one expanding arc $e \in T$.

Proof: By the above remark we assume the action of $Fix(\phi)$ on $T_{Fix(\phi)}$ is hyperbolic and we show there is an expanding edge $e \in T$. A hyperbolic action of a group on a tree is characterized by the existence of two elements acting hyperbolically on the tree with disjoint axis [CM]. Let $g, h \in Fix(\phi)$ be these hyperbolic elements with disjoint axis A_g and A_h. Let $\overline{a_ga_h}$ be the geodesic connecting A_g and A_h.

Following Construction 4.8 we can construct ϕ_* from ϕ. If v_g is a vertex on A_g, since $\phi(g^n) = g^n$ and ϕ_* was constructed to be G equivariant, we have $\phi_*(g^n(v_g)) =$

$g^n(\phi_*(v_g))$. By choosing an appropriate n we may assume $g^n(v_g) \in \overline{a_g\phi_*(g^n(v_g))}$. Let $v = g^n(v_g)$. Likewise on A_h we find a w satisfying similar conditions. It is then the case that $\overline{vw}$ is an expanding arc, and by Lemma 4.10 we are done. ♠

Lemma 4.14: *Suppose $\phi \in Aut(G)$ and that the action of $Fix(\phi)$ on $T_{Fix(\phi)}$ is hyperbolic. Let f be an edge in $T_{Fix(\phi)}/Fix(\phi)$, $\tilde{f}$ a lift of f in $T_{Fix(\phi)}$, and $Stab(\tilde{f}) =< g >$. It is the case that $\phi(g) = g$ or $\phi(g) = g^{-1}$.*

Proof: Following Construction 4.8 we obtain ϕ_* from ϕ. By Lemma 4.13, under ϕ_*, T contains and length one expanding edge e. A theorem of Bass [B] tells us that the minimal tree in a group action is the union of the axis of hyperbolically acting elements. Hence there is a $h \in Fix(\phi)$ acting hyperbolically on $T_{Fix(\phi)}$ with axis containing $\tilde{f}$. Since ϕ_* was constructed to be G equivariant $h^n(e)$ are expanding edges for all n. Without loss of generality we may assume that there are integers n and m such that $\tilde{f}$ is contained in the geodesic connecting $h^n(e)$ and $h^m(e)$. Let E be the smallest arc containing $h^n(e)$ and $h^m(e)$. Since $h^n(e)$ and $h^m(e)$ are length one expanding arcs, E is also an expanding arc.

Suppose there are no two adjacent edges on E with trivially intersecting edge stabilizers. If this were the case then by Lemma 4.4 there would be an element $k \in G$ which stabilizes all of E. Since E is an expanding arc it contains an expanding edge which has a stabilizer containing k. By Lemma's 4.3 and 4.11 $\phi(k) = k^{\pm 1}$. But k also stabilizes $\tilde{f}$. Hence $k = g^n$ for some n, and by Lemma 4.3 $\phi(g) = g^{\pm 1}$.

We will show that if e_1 and e_2 are adjacent edges on E whose stabilizers intersect trivially, then ϕ_* fixes $e_1 \cap e_2$. If this is the case then we can pick a sub arc of E, which contains $\tilde{f}$, is expanding, and contains no two adjacent edges with trivially intersecting edge stabilizers. We are then done by the preceding paragraph.

Consider the sub arcs E_i of E determined by those vertices of E which lie between adjacent edges whose stabilizers intersect trivially. Again by Lemma 4.3 for each E_i there is a $g_i \in G$ which stabilizes all of E_i. Further, if $i \neq j$ then $< g_i > \cap < g_j >$ is the trivial group. This implies that if $i \neq j$ $\phi_*(E_i) \cap \phi_*(E_j)$ is at most a point. Also $\phi_*(E_i)$ can't intersect more than one E_j. These observations along with the fact that E is an expanding arc implies that ϕ_* must fix the endpoints of each E_i. ♠

We are now ready for the proof of Lemma 2.

Proof of Lemma 2: Choose an automorphism ϕ of G. Suppose $Fix(\phi)$ is not trivial or infinite cyclic. Consider the action of $Fix(\phi)$ on $T_{Fix(\phi)} = (T_{Per(\phi)})_{Fix(\phi)}$, where T is a JSJ tree for G promised by Theorem 1.8. Clearly we have (i) since T is a tree for G which has infinite cyclic edge stabilizers.

Suppose a JSJ decomposition for G has n edge groups. Further suppose that m is the maximum index of all edge groups in their respective normalizers. It is shown in [S] that both n and m are invariants of G, and hence so is nm. We claim that over all automorphisms ϕ, $T_{Fix(\phi)}/Fix(\phi)$ has at most $2nm$ edges.

Suppose $T_{Fix(\phi)}/Fix(\phi)$ has greater than $2nm$ edges. We claim it is then the case that there are $2m+1$ edges $e_0, e_1, ..., e_{2m}$ in $T_{Fix(\phi)}/Fix(\phi)$, with lifts $\tilde{e_i}$ for $i = 0, 1, 2, ..., 2m$ in $T_{Fix(\phi)}$, which are in the same orbit under $Per(\phi)$. Since there are only n G orbits of edges clearly there are $2m+1$ lifts which are G equivalent. Since $T_{Fix(\phi)} = (T_{Per(\phi)})_{Fix(\phi)}$ all of the lifts are in $T_{Per(\phi)}$. Since $Per(\phi)$ is a $\mathbb{Z} - factor$ of G, any edges in $T_{Per(\phi)}$ which are G equivalent are also $Per(\phi)$ equivalent. Hence we have our claim.

We force a contradiction by showing two of these edges are in the same orbit under $Fix(\phi)$. Suppose γ stabilizes e_0. By the last paragraph there exist g_i in $Per(\phi)$ with $g_i\gamma g_i^{-1}$ stabilizing e_i for $i = 1, 2, ..., 2m$. By Lemma 4.14 we know that $\phi(\gamma) = \gamma$ or $\phi(\gamma) = \gamma^{-1}$, and for each i we have $\phi(g_i\gamma g_i^{-1}) = g_i\gamma g_i^{-1}$ or $\phi(g_i\gamma g_i^{-1}) = g_i\gamma^{-1}g_i^{-1}$

If $\phi(\gamma) = \gamma$ then $\phi(g_i\gamma g_i^{-1}) = \phi(g_i)\gamma\phi(g_i^{-1})$, which we claim must equal $g_i\gamma g_i^{-1}$. If $\phi(g_i\gamma g_i^{-1}) = g_i\gamma^{-1}g_i^{-1}$ then $g_i^{-1}\phi(g_i)$ is an element which conjugates γ to its inverse, something that can't happen in a torsion free hyperbolic group. Since $\phi(g_i)\gamma\phi(g_i^{-1}) = g_i\gamma g_i^{-1}$, $g_i^{-1}\phi(g_i)$ normalizes γ and hence $\phi(g_i) = g_i\overline{\gamma}^{r_i}$, where $\overline{\gamma}$ generates the normalizer of γ and r_i is some integer. Since $g_i \in Per(\phi)$ r_i must be zero and ϕ must fix each g_i. This implies all of the edges are in fact in the same $Fix(\phi)$ orbit, contradicting the fact that they are lifts of different edges in $T_{Fix(\phi)}/Fix(\phi)$.

If $\phi(\gamma) = \gamma^{-1}$ by the same reasoning as above we have $\phi(g_i\gamma g_i^{-1}) = g_i\gamma^{-1}g_i^{-1}$ and $\phi(g_i) = g_i\overline{\gamma}^{r_i}$. Recall that m is the maximum index of the edge groups in their respective normalizers. Hence we have $\overline{\gamma}^{m_\gamma} = \gamma$ for some m_γ whose absolute value is less than or equal to m. Notice that $\phi(g_i\gamma^r) = g_i\overline{\gamma}^{r_i}\gamma^{-r} = g_i\gamma^r\overline{\gamma}^{r_i}\gamma^{-2r} = (g_i\gamma^r)\overline{\gamma}^{r_i - 2m_\gamma r}$. Choosing r appropriately (a different one for each g_i) we may assume $0 \leq r_i - 2m_\gamma r \leq |2m_\gamma|$. Replace each g_i with $g_i\gamma^r$ (where the r is chosen for each i to insure that the last inequality holds) and still call it g_i. This new g_i still stabilizes e_i. Since $|2m_\gamma| \leq 2m$ and there exist g_i and g_j, with i not equal to j with (using the new values) $r_i = r_j$. It is then the case that $g_ig_j^{-1}$ is fixed by ϕ and takes e_j to e_i. Contradicting the fact that the e_i are lifts of different edges in $T_{Fix(\phi)}/Fix(\phi)$. This completes the proof of (ii).

Suppose $H \subset Fix(\phi)$ is not contained in an infinite cyclic group and stabilizes a vertex $v \in T$. It's then the case that $\phi_*(v) = v$ This implies ϕ restricts to an automorphism of the stabilizer of v in G. This shows (iii). ♠

CHAPTER 5

In this chapter we prove Lemma 3. We assume throughout this chapter that G is a torsion free, freely indecomposable hyperbolic group. Before we are ready for the proof of Lemma 3 we need a few observations pertaining to a certain short exact sequence.

Since a torsion free hyperbolic group which is not infinite cyclic has no center, it is the case that the identification of it with its groups of inner automorphisms is an isomorphism. Let V be a JSJ vertex group for G which is not quadratically hanging and not infinite cyclic. Since every $\phi \in Inn(V)$ can be extended to an automorphism of G, by identifying V with $Inn(V)$ we obtain the following short exact sequence:

$$1 \to V \to Aut_G(V) \to K \to 1.$$

Here $K \subset Out(V)$ and by Proposition 1.13 is finite. For $\phi \in Aut_G(V)$ its action as an automorphism of V is realized by its action as conjugation in $Aut_G(V)$ restricted to $Inn(V)$. That is to say, if i is the identification of V with $Inn(V)$ and $g \in V$ then $\phi(g) = i^{-1}(\phi i(g)\phi^{-1})$. Notice that $\phi i(g)\phi^{-1} \in Inn(V)$ since $Inn(V)$ is normal in $Aut_G(V)$.

If $\phi \in Aut_G(V)$ denote by $Fix_V(\phi)$ its fixed subgroup in V. Denote by i_ϕ the inner automorphism of $Aut_G(V)$ determined by ϕ. We are concerned with the different possibilities, over $Aut_G(V)$ for $Fix_{Aut_G(V)}(i_\phi) \cap Inn(V)$. The following lemma will be useful:

Lemma 5.1: *Let ρ and ϕ be elements of $Aut_G(V)$. It is the case that $\rho(Fix_{Aut_G(V)}(i_\phi))\rho^{-1} = Fix_{Aut_G(V)}(i_{\rho\phi\rho^{-1}})$.*

Proof: This lemma is the fact that the conjugate of the centralizer is the centralizer of the conjugate. First we show containment from left to right. Let $h \in Fix_{Aut_G(V)}(i_\phi)$ and consider $i_{\rho\phi\rho^{-1}}(\rho h \rho^{-1})$. This is equal to $\rho\phi\rho^{-1}\rho h\rho^{-1}\rho\phi\rho^{-1}$. Canceling the $\rho^{-1}\rho$ and remembering $h \in Fix_{Aut_G(V)}(i_\phi)$ we see we have $\rho h \rho^{-1}$.

For the other direction we show that for $h \in Fix_{Aut_G(V)}(i_{\rho\phi\rho^{-1}})$, $\rho^{-1}h\rho \in Fix_{Aut_G(V)}(i_\phi)$. If $h \in Fix_{Aut_G(V)}(i_{\rho\phi\rho^{-1}})$ then $\rho\phi\rho^{-1}h\rho\phi\rho^{-1} = h$, which implies $\phi\rho^{-1}h\rho\phi = \rho^{-1}h\rho$. The left side of the last equality is $i_\phi(\rho^{-1}h\rho)$ and is hence fixed by i_ϕ, as was to be shown. ♠

Recall the definitions of $\mathcal{U}$ and $\mathcal{V}$ in section 2.

Lemma 3: *It is the case that $|\mathcal{V}/\sim|$ is finite.*

Proof: By Lemma 2 (iii) vertex groups in $\mathcal{U}$ are infinite cyclic or arise as the restriction of automorphisms of G to vertex stabilizers in some JSJ tree.

Suppose V is a quadratically hanging vertex group. By Theorem 1.11 and [J] fixed subgroups correspond to the fundamental groups of essential sub surfaces. Homological considerations imply that if the genus of the underlying surface is k then such groups are free with rank at most $2k$. Hence up to isomorphism there are only finitely many.

If V is not a quadratically hanging subgroup consider the short exact sequence described above. Since V is hyperbolic and is finite index in $Aut_G(V)$, $Aut_G(V)$ is hyperbolic.

If $\phi \in Aut_G(V)$ is not a torsion element then a theorem of Gromov [G] states that $Fix_{Aut_G(V)}(i_\phi)$ is virtually infinite cyclic. The intersection of a virtually infinite cyclic group and the torsion free group $Inn(V)$ is contained in an infinite cyclic group.

If $\phi \in Aut_G(V)$ is torsion then another theorem of Gromov states that there are only a finite number of conjugacy classes of such elements. Hence it suffice to prove our claim over one such conjugacy class. By Lemma 5.1 each conjugacy class of $Aut_G(V)$ determines, by acting as inner automorphisms of $Aut_G(V)$, fixed subgroups which are conjugate in $Aut_G(V)$. We are concerned with the intersection of these subgroups with $Inn(V)$. Since $Inn(V)$ is normal in $Aut_G(V)$ the intersection of each these fixed subgroups with $Inn(V)$ are isomorphic.

Hence up to isomorphism (not respecting conjugacy classes of possible edge groups) there are only finitely many possibilities for vertex groups.

To finish the proof let $V_1, V_2, ..., V_n$ be the JSJ vertex groups of G stabilizing the vertices $v_1, v_2, ..., v_n$ of T. Suppose $(H, L) \in \mathcal{V}$ with ϕ being an automorphism of G which produces this pair. If H stabilizes the vertex $v \in G$ then there is a $g \in G$ with $g(v) = v_i$ for some i. It is then the case that the automorphism $i_g \phi i_g^{-1}$ produces a pair $(H_*, L_*) \in \mathcal{V}$ with H_* stabilizing v_i and $(H, L) \sim (H_*, L_*)$. Hence we only need to look at the pairs in $\mathcal{V}$ where the fixed vertex group stabilizes one of the v_i. Since there are only finitely many such v_i we show our claim for just one such vertex.

Suppose V_i is not a quadratic hanging subgroup. We induct on the number of conjugacy classes in L. We note that if there are none we are done by the first paragraphs of this proof. Let $< g_1 >, < g_2 >, ..., < g_m >$ be the JSJ edge groups incident to V_i. So suppose we are looking at pairs (H, L) with L containing one conjugacy class of an element in H. By constructing an new automorphism, as in the last paragraph, we may assume that we are looking at all pairs (H, L) with L containing some g_i. It suffices then to show our claim for all pairs (H, L) with L containing, say, g_i. The set of $\phi \in Aut_G(V_i)$ fixing g_i clearly form a subgroup $N \subset Aut_G(V_i)$. In fact this subgroup is virtually infinite cyclic but we will not need this. If $\phi \in N$ and is non torsion then, as was seen above, $Fix_{V_i}(\phi)$ is an infinite cyclic groups containing $< g_i >$. Since normalizers are infinite cyclic there are only finitely many such groups in a hyperbolic group. It is the case that there are finitely many conjugacy classes of torsion elements in N. Suppose ϕ is a torsion element of N and that $\rho \in N$. If $< g_i > \in Fix_{V_i}(\phi)$ and $< g_i > \in Fix_{V_i}(\rho\phi\rho^{-1})$ then by Lemma 5.1 the automorphism i_ρ of $Aut_G(V_i)$ provides an abstract isomorphism of $Fix_{V_i}(\phi)$ to $Fix_{V_i}(\rho\phi\rho^{-1})$ which fixes $< g_i >$. Hence the two pairs of $\mathcal{V}$ determined by ϕ and $\rho\phi\rho^{-1}$ are equivalent. If L contains n conjugacy classes then we again construct a new automorphism so that L contains g_i. By the case $n = 1$ there are only finitely many such pairs (up to our equivalence) which contain g_i and hence still only finitely many if we enlarge L. This concludes our induction and the case that V_i is not a quadratic hanging subgroup.

If v_i is stabilized, in G by a quadratically hanging subgroup then by Theorem 1.11 the subgroup where an automorphism is periodic (if the fixed subgroup is not contained in an infinite cyclic group) is realized by the fundamental group of an essential sub surface. By [J] the fixed subgroup of a periodic automorphism is infinite cyclic, or realized by the fundamental group of an essential sub surface. Which implies the fix subgroup of an arbitrary automorphism is fundamental group of an essential sub surface. If the fixed subgroup of some automorphism has a subgroup which stabilizes such a vertex along with some adjacent edge groups then the underlying fixed subgroup corresponds to an essential subsurface which contains the boundary curves associated to those edge groups. It is the case that there are finitely many, up to homeomorphism fixing those boundary curves, such sub surfaces. These homeomorphisms induce isomorphisms and we are done. ♠

Corollary: *Suppose G is a torsion free hperbolic group which does not split over the trivial or infinite cyclic group. There are only finitely many, up to conjugation, fixed subgroups which are not isomorphic to the infinite cyclic group.*

Proof: Again consider the above short exact sequence. If an automorphism is non torsion (such as conjugation) its fixed subgroup is trivial or infinite cyclic. By lemma 5.1 there are finitely many fixed subgroup coming from torsion elements, where conjugation is in $Aut(G)$. Since G is finite index in $Aut(G)$ we see there are still only finitely many such fixed subgroups up to conjugation in G. ♠