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DIFFERENTIABILITY AND TOPOLOGY OF LABYRINTHS IN THE DISC AND ANNULUS

GILBERT LEVITT and HAROLD ROSENBERG

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§0. INTRODUCTION

IT IS WELL-KNOWN that vector fields on the disc, or more generally on a planar surface, do not exhibit complicated global phenomena: by the Poincaré–Bendixson theorem, every recurrent orbit is either a singularity or a simple closed curve.

The situation is completely different for line fields. In particular, it was shown in [6] that it is possible for an orbit of a line field to lose itself in a labyrinth, i.e. to enter a disc and stay inside for ever without either going to a singularity or spiralling towards a compact cycle.

Line fields in the disc and the annulus were studied and classified in [6], under the assumption that the singularities are thorns and tripods. It was also asked in [6] to what extent line fields with nontrivial recurrence can be differentiable.

We intend in this paper first to partially answer this question, then to extend the classification in [6] to the case when singularities are thorns and saddles with any number of prongs.

First we discuss differentiability. There are several natural definitions of C^r-differentiability for singular line fields (even for orientable ones). First one can require the line field to be C^r in the complement of the singularities; it was pointed out in [6] that being C^∞ in this sense does not restrict the dynamics of the foliation.

Another notion of differentiability imposes strong C^r local models near the singularities (see §2). Assuming C^2-differentiability in this sense, we prove that the Poincaré–Bendixson theorem continues to hold; that is, every orbit entering a disc and staying inside has to go to a singularity or spiral towards a compact cycle. The proof is inspired in part by Denjoy's theorem about C^2 vector fields on the torus.

This is in sharp contrast with the fact due to Cherry [1], that on the torus minus a disc there are vector fields (C^∞ in the above sense) where every orbit coming in from the boundary stays inside forever and does not spiral towards a compact cycle.

A very interesting question is whether the Poincaré–Bendixson theorem continues to hold for line fields with hyperbolic-type singularities; by this we mean that the natural holonomy maps obtained by following a separatrix through a singularity are of the same form as for hyperbolic vector fields.

In §3 and §4 of this paper, we extend the results of [6] to line fields with thorns and n-prong saddles. For the disc, the situation is not essentially different from the case of thorns and tripods. However, for the annulus, the study is quite different and more complicated. In both cases, we obtain a topological structure theorem for the foliation. In several instances the proofs we give simplify arguments in [6].

§1. PRELIMINARIES

We consider foliations of compact surfaces, whose singularities are thorns and saddles with any number $p \geq 3$ of prongs.

We define an *arational* foliation as in [6]: $\mathscr{F}$ is arational if $\mathscr{F}$ has no interior compact leaf, $\mathscr{F}$ is transverse to δM, and no separatrices join two singularities.

We define the *spreading* of a leaf as in [6, p. 3], but we allow the spreading of a singular leaf (see Fig. 1); this operation replaces a p-prong saddle by a $(p+1)$-prong saddle and a thorn. If x and y are points on the same leaf, we denote by (x, y) the leaf joining x to y. We also use this notation when one of the points is a singularity and the leaf of the other point is a separatrix of the singularity.

A *standard foliation* of the half-disc D^+ is defined as in [6, p. 4], by consecutively spreading a finite number of leaves in the foliation of D^+ by concentric circles, but we allow the top point of D^+ to be a saddle (see Fig. 2).

A *labyrinth* over a segment J (resp. a simple closed curve C), and its *standard extension* to the disc D^2 (resp. the annulus A), are defined as in [6, pp. 4–5].

For future reference, we note the following useful fact: in an arational foliation, every leaf is cut by a transverse curve, and every infinite leaf (i.e. leaf of infinite length) is cut infinitely often by some transverse curve (compare [6, p. 16]);unless otherwise indicated, transverse curves will always be simple and closed.

Also note that the extension lemma [6, lemma 2.2, p. 9] remains valid in our situation.

§2. DIFFERENTIABILITY

There is no obvious definition of differentiability for a singular foliation of a surface M.

One possibility is to say that $\mathscr{F}$ is of class C^r if the restriction of $\mathscr{F}$ to $M - \operatorname{Sing} \mathscr{F}$ is defined by a C^r line field.

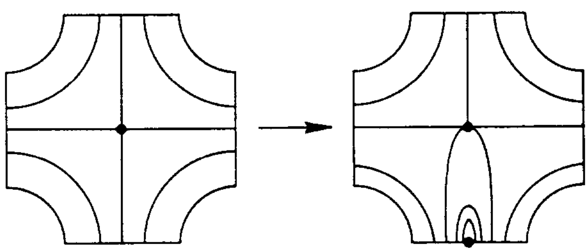

Fig. 1.

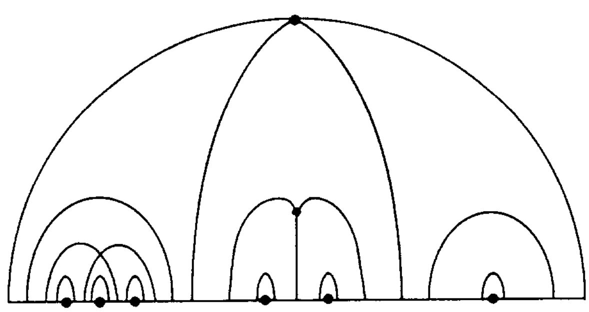

Fig. 2.

With this definition, Schwartz's theorem [7] implies that in a singular C^2 foliation every compact invariant set contains a compact leaf or a singularity. Conversely, Gutierrez's theorem [3] makes it likely that, if every compact invariant set contains a compact leaf or a singularity, then $\mathscr{F}$ is topologically conjugate to a C^∞ foliation (this is proved in [3] for orientable foliations, and in [6, pp. 31–32] for a special family of labyrinths).

We get another natural definition by imposing *local models* near the singularities. We shall say from now on that *$\mathscr{F}$ is of class C^r* if there exists a C^r-atlas $\{\Phi_\alpha : U_\alpha \to D_\alpha\}$, where each Φ_α carries the chart domain U_α foliated by $\mathscr{F}$ to a subset D_α of the complex plane foliated by the level sets of the function $|\mathrm{Re}\, z^{k/2}|$ (k is a positive integer, depending on α; if k is odd, the absolute value sign removes the ambiguity in the definition of $z^{k/2}$; if $0 \in D_\alpha$, then U_α contains a thorn if $k = 1$, a k-prong saddle if $k \geq 3$). Note that, if $\mathscr{F}$ is of class C^r, then in the extension lemma (lemma 2.2 of [6]) the holonomy map extends to a C^r map defined on a compact interval.

This notion of differentiability imposes surprisingly strong restrictions on the global behavior of $\mathscr{F}$.

THEOREM 1. *Let $\mathscr{F}$ be an arational foliation of the disc or annulus. If $\mathscr{F}$ is of class C^2, then every regular leaf is compact (and goes from boundary to boundary).*

We refer the reader to [5] for a more general discussion of differentiability and corollaries of Theorem 1.

Proof of Theorem 1. The proof is a generalization of a well-known argument of Denjoy [2, 8]. First we note that it suffices to show that regular leaves meeting δM are compact: if this is true, then the union of compact regular leaves and separatrices reaching δM is nonempty and open in $M - \mathrm{Sing}\, \mathscr{F}$, but by the extension lemma it is also closed.

We assume that there is a point $m \in \delta M$ whose leaf l is regular and does not reutrn to δM, and we argue towards a contradiction. We orient l, starting from m.

Using a transverse curve C meeting l infinitely often, we construct a transverse segment H, with endpoints a and b on δM, which meets l infinitely often (see Fig. 3(i)). From now on we shall picture H as horizontal (see Fig. 3(ii)); it separates M into two regions (top and bottom).

Consider the limit set of l (equal to $\bar{l} - l$). It is compact and meets H. Let t be the point of $(\bar{l} - l) \cap H$ furthest to the left.

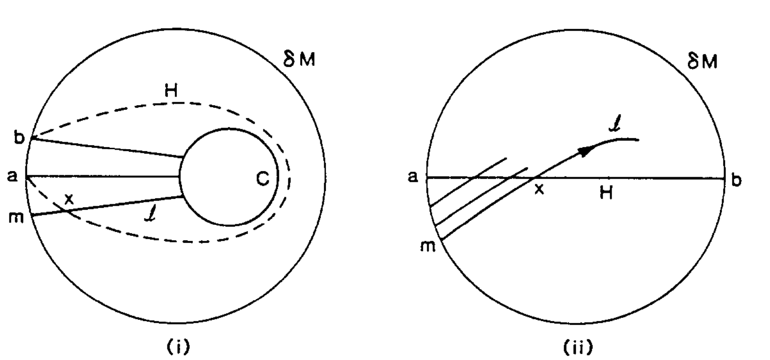

Fig. 3.

LEMMA 1. *l meets the interval $[a, t) \subset H$ finitely many times; in other words, l accumulates on t only from the right.*

This lemma is a special case of a general fact, valid for any arational foliation: if an open transverse interval does not meet the limit set of a half-leaf l, then it meets l finitely many times.

Proof of Lemma 1. Consider the points of $[a, t)$ where a separatrix first meets $[a, t)$, for instance a' and a'' on Fig. 4. There are finitely many of them, and we call a' the furthest to the right (see Fig. 4); $a' = a$ if $[a, t]$ meets no separatrix. We claim that l meets (a', t) at most once.

Suppose on the contrary that y and y' are successive intersections of l with (a', t). First note that l goes in the same direction (up or down) at y and y': if not, the extension lemma would show that some separatrix meets $[a, t)$ for the first time between y and y'.

We now apply the extension lemma to the segment of l between y and y', and conclude that either the leaf through t meets H on the left of t, or l spirals towards a compact leaf or a union of singularities and separatrices. Both conclusions are absurd. This proves Lemma 1. □

Let x_0 be the point of $l \cap [a, t)$ furthest to the right (x_0 exists because $l \cap [a, t)$ is finite and contains x, see Fig. 3(ii) or 4). Following l from x_0 on, we call $x_1, \ldots, x_n, \ldots$ its successive intersections with H.

Consider the integers p such that the interval $[x_0, x_p] \cap H$ contains t but contains no x_n, $0 < n < p$ (see Fig. 5, where $p = 1, 2, 3, 8, 9$, etc.). These integers form an infinite sequence whose terms are alternatively odd and even, and we fix an odd $p = 2q + 1$. Note that the points x_i $(1 \leq i \leq p-1)$ with odd and even subscripts *alternate* on H.

We shall reach a contradiction, for p large enough, by considering on H a partially defined second return map. Assume that at x_0 the leaf l is going up (as on Fig. 5); then we define the *downward second return map* f, by saying that $f(y)$ is the first point where the half-leaf leaving y downward returns to H from the top (for instance, on Fig. 5, $f(x_0) = x$, $f(x_1) = x_3$, $f(x_2) = x_0$; $f(x)$ is not defined). If l is going down at x_0, we consider the upward return map instead.

The domain of definition U of f is a finite nonempty collection of open intervals (at whose endpoints the half-leaf reaches a singularity before meeting H twice). The map f is an orientation-preserving C^2-diffeomorphism between U and $f(U)$. Our definition of differentiability implies furthermore that *the logarithm of the derivative of f has bounded variation on U*:

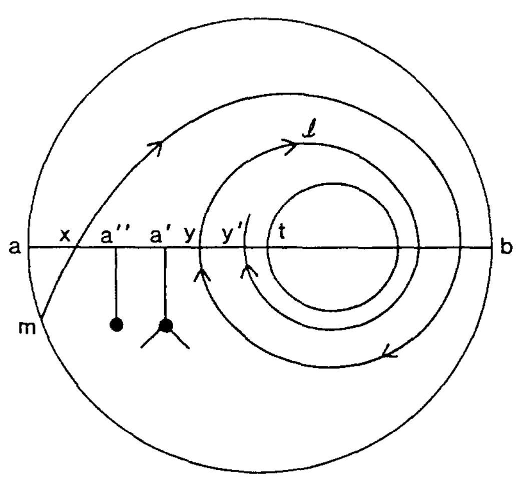

Fig. 4.

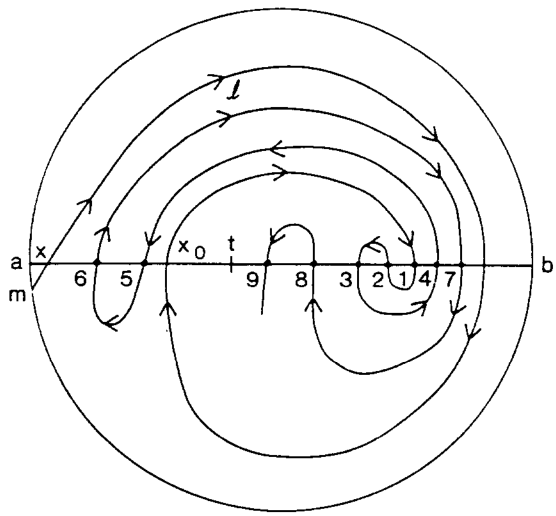

Fig. 5. We write 1, 2, . . . instead of $x_1, x_2, \ldots$.

there is a number V such that, for any set $y_0 < y_1 < \ldots < y_{2h-1}$ on U, we have

$$\left|\sum_{i=0}^{2h-1} (-1)^i \log Df(y_i)\right| < V.$$

Let $I \subset \delta M$ be an open interval containing m and meeting no separatrix. Leaves starting at points of I stay close to l, and for every n they cut out an open interval $I_n \subset H$ around x_n. The intervals I_n are disjoint, and their length $|I_n|$ tends to 0 as n goes to infinity.

We have $I_p = f^q(I_1)$ and $I_0 = f^q(I_{p-1})$, so we can find points $y \in I_1$ and $z \in I_{p-1}$ such that $|I_p| = Df^q(y)\cdot|I_1|$ and $|I_0| = Df^q(z)\cdot|I_{p-1}|$. Using the chain rule for derivatives, we obtain

$$\begin{aligned}\log(|I_p|\cdot|I_{p-1}|/|I_0|\cdot|I_1|) &= \log Df(y) + \log Df(f(y)) + \ldots + \log Df(f^{q-1}(y)) \\ &\quad - \log Df(z) - \log Df(f(z)) - \ldots - \log Df(f^{q-1}(z)).\end{aligned}$$

The points $y, f(y), \ldots, f^{q-1}(y)$ belong to the intervals $I_1, I_3, \ldots, I_{p-2}$; the points z, $f(z), \ldots, f^{q-1}(z)$ belong to $I_{p-1}, \ldots, I_4, I_2$. The alternating property pointed out earlier then implies that the absolute value of $\log(|I_p|\cdot|I_{p-1}|/|I_0|\cdot|I_1|)$ is less than V, a contradiction for p large enough. □

§3. ARATIONAL FOLIATIONS OF THE DISC

Just as in [6], we first prove the following.

THEOREM 2. *Let $\mathscr{F}$ be an arational foliation of D^2 such that no leaf joins a point of δD^2 to a thorn. Then $\mathscr{F}$ is the standard extension to D^2 of a labyrinth on a segment J.*

(1) The first step in the proof is to *construct an arc J passing through all the thorns and transverse to $\mathscr{F}$* (except at the thorns). The proof we give is simpler than the one in [6, pp. 10–13].

LEMMA 2. *Let $\mathcal{F}$ be an arational foliation of a compact surface M. Any two points x and y in $M - \text{Sing}\,\mathcal{F}$ can be joined by an immersed transverse arc.*

(In fact we can join x and y by an embedded arc, but we do not need this.)

Using Lemma 2, we construct an immersed J passing through all the thorns. We then replace it by an embedded arc, as in [6, pp. 13–15].

Proof of Lemma 2. Fix x, and consider the subset V of $M - \text{Sing}\,\mathcal{F}$ consisting of all the points that can be joined to x by a transverse (immersed) arc. This subset is obviously nonempty and open. We claim that it is also closed (and therefore contains y).

Let z be a point in the frontier of V, and C a transverse closed curve meeting the leaf of z. For $z' \in V$ close enough to z, we can extend to z any transverse arc joining x to z' (see Fig. 6). Thus $z \in V$. □

(2) Let J be as above. *Any half-leaf that does not go to a singularity goes to J or δD. If a separatrix from a saddle s goes to δD* (without meeting J), *all the other separatrices of s go to J. No regular leaf goes from δD to δD without meeting J.*

Same proof as in [6] (proof of 2.4, p. 16), using the fact that a p-prong saddle is a singularity of negative index $(2-p)/2$.

(3) Let $x_1, \ldots, x_n$ be the points of δD whose leaves go to saddles $y_1, \ldots, y_n$ (before intersecting J). Note that there is at least one such point: consider a segment of a leaf going from δD to J, and use the extension lemma [6, p. 9] in order to lift it until reaching a singularity; since no leaf joins a point of δD to a thorn, this singularity is a saddle.

The end of the proof of Theorem 2 is as in [6, pp. 16–20], using for each y_i the two separatrices adjacent to the separatrix (y_i, x_i). □

THEOREM 3. *Let $\mathcal{F}$ be an arational foliation of D^2. Then $\mathcal{F}$ is obtained by a finite number of consecutive spreadings from either:*

(a) *the trivial arational foliation of D^2 with exactly two thorns, or*

(b) *the standard extension of a labyrinth L on a segment J.*

The spreadings are done on (possibly singular) leaves going to δD^2.

Proof of Theorem 3. The proof is by induction on the number q of leaves joining a thorn to δD^2. We know that the theorem is true for $q = 0$.

Now suppose that a leaf joins a thorn z to a point $x \in \delta D^2$. Apply the extension lemma to leaves near (z, x). If $\mathcal{F}$ is not the trivial foliation with two thorns, one arrives at a saddle s (see Fig. 7).

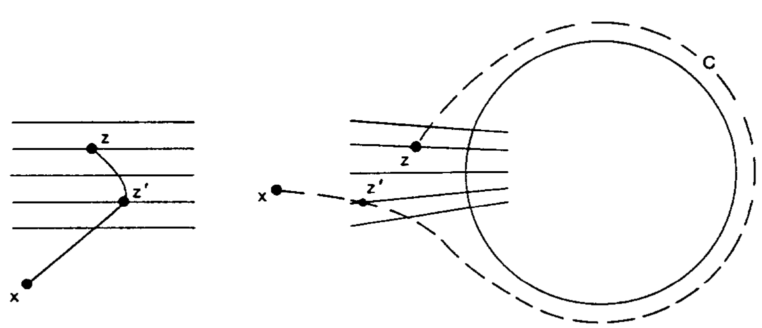

Fig. 6.

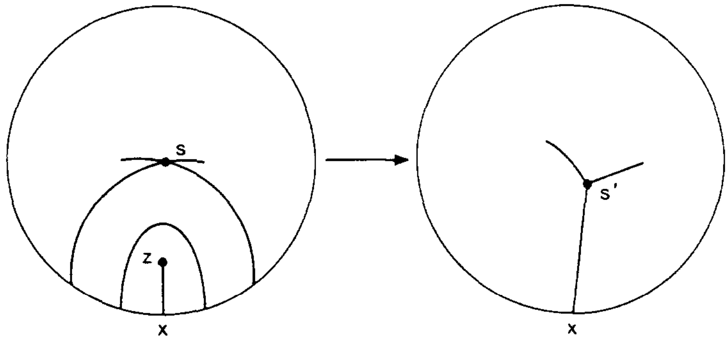

Fig. 7.

Modify the foliation as in Fig. 7, replacing s by a saddle s' with one less prong (a regular point if s is a tripod). The result now follows by applying the induction hypothesis to the new foliation $\mathscr{F}'$; one passes from $\mathscr{F}'$ to $\mathscr{F}$ by spreading the leaf (s', x). □

§4. ARATIONAL FOLIATIONS OF THE ANNULUS

For foliations with p-prong saddles, it turns out that the situation on the annulus A is much more complicated than on the disc. We start with an example of an arational foliation of A which cannot be obtained by spreading the product foliation or the standard extension of a labyrinth on a simple closed curve C.

Say that a singularity s (saddle or thorn) is *joined to the boundary* if at least one separatrix of s reaches δM.

Example 1

Suppose $\mathscr{G}$ is an arational foliation of D^2 such that no thorn and exactly one saddle s is joined to δD^2. Consider a punctured annulus (or pair of pants) with the foliation pictured in Fig. 8, and fill in D with D^2 equipped with $\mathscr{G}$, in such a way that x is identified with a point of δD^2 whose leaf goes to s.

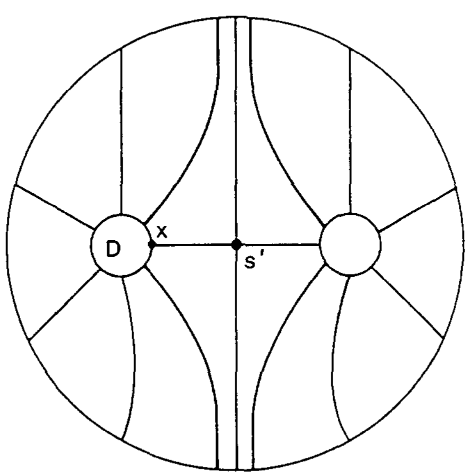

Fig. 8.

After collapsing to a point s'' the saddle connection (s, x, s'), we get an arational foliation $\mathscr{F}$ of A with the property that exactly one singularity (namely s'') is joined to δA. Such a foliation cannot be the standard extension of a labyrinth, since in the extension of a labyrinth there are at least two saddles joined to δA (one on each side of C).

Example 2

Here is another example. Suppose $\mathscr{G}$ is an arational foliation of D^2 such that no thorn and exactly one saddle s_1 (resp. exactly two saddles s_1 and s_2) is (are) joined to δD^2. Glue D^2 to a punctured annulus equipped with the foliation pictured in Fig. 9, in such a way that x is identified with a point of δD^2 whose leaf goes to s_1, and y to a point of δD^2 whose leaf is regular (resp. whose leaf goes to s_2).

Then collapse to a point the saddle connection(s) (s_1, x, s') (resp. (s_1, x, s') and (s_2, y, s')). Just as above, the resulting foliation on A is not the standard extension of a labyrinth.

THEOREM 4. *Let $\mathscr{F}$ be an arational foliation of the annulus A. Then $\mathscr{F}$ is obtained by a finite number of spreadings of either*

—the product foliation, or

—the standard extension of a labyrinth on a simple closed curve C, or

—a foliation obtained from D^2 as above (examples 1 and 2).

The rest of the section is devoted to the proof. Note that we need only consider the case when no thorn is joined to δA: the extension to the general case is as above (Proof of Theorem 3).

We first remark that, if $\mathscr{F}$ is not the product foliation, each component C_i of δA contains a point whose leaf goes to a saddle: apply the extension lemma to a segment of leaf joining a point $x \in C_i$ to another point of δA or to a point belonging to a transverse curve meeting the leaf of x infinitely often.

If *no singularity* is joined to δA, then $\mathscr{F}$ is the product foliation. Now assume that *exactly one singularity* s is joined to δA, and consider two separatrices (s, a) and (s, b) joining s to different components of δA.

First suppose these two separatrices are adjacent. Then another separatrix, adjacent to (s, a) or (s, b), must also go to δA (see Fig. 10(i)), and $\mathscr{F}$ is obtained as in Example 1 (note that the dotted curve on Fig. 10(ii) is transverse to the foliation).

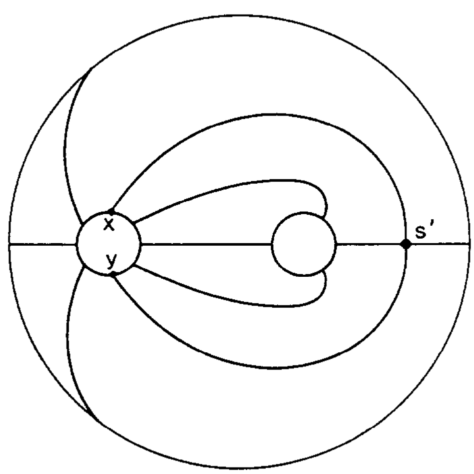

Fig. 9.

Now suppose (s, a) and (s, b) are not adjacent (see Fig. 11). Then $\mathscr{F}$ is obtained as in Example 2; the foliation $\mathscr{G}$ has one (resp. two) saddle(s) joined to δD^2 if there is (resp. there is not) a separatrix adjacent to both (s, a) and (s, b).

From now on we assume that there are *two distinct saddles* s_1 *and* s_2 joined to points m_1 and m_2 of δA. We will show that $\mathscr{F}$ is the standard extension of a labyrinth.

The hard part of the proof is to *find a noncontractible simple closed curve C passing through all the thorns and transverse to* $\mathscr{F}$ (except at the thorns). Once we have C, the end of the proof is as in [6, p. 24].

Just as above (first step of the proof of Theorem 2), we can construct *an embedded segment J transverse to* $\mathscr{F}$ *and passing through all the thorns*. In fact, we also need J to be *disjoint from* (s_1, m_1) *and* (s_2, m_2). The proof that we can find such a J will be postponed until the end of the paper.

From now on, we will say that a separatrix *goes to* δA only when it does so *without meeting* J.

As in [6, top of p. 22] it is easy to see that every half-leaf which does not go to a singularity reaches J or δA. Also note that, if (s, a) and (s, b) are separatrices going to the same component of δA (without meeting J), then the foliation is "trivial" in the annulus bounded by (s, a) and (s, b) (see Fig. 12), and every separatrix between (s, a) and (s, b) goes to J.

We claim that *there is a saddle with a separatrix going to J and an adjacent separatrix going to* δA; the proof we are about to give is simpler than the proof of step 2 in [6, pp. 22–23].

In $M - \operatorname{Sing} \mathscr{F}$, consider the union of all leaves meeting δA, and the union of all leaves meeting J. These two sets are open, and we know that they cover $M - \operatorname{Sing} \mathscr{F}$. It follows that

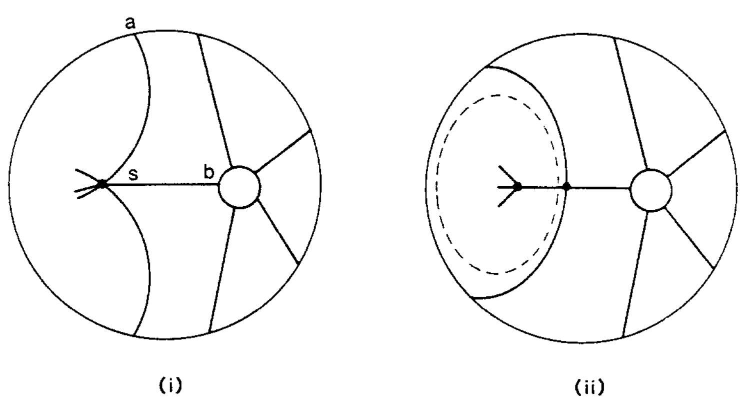

Fig. 10.

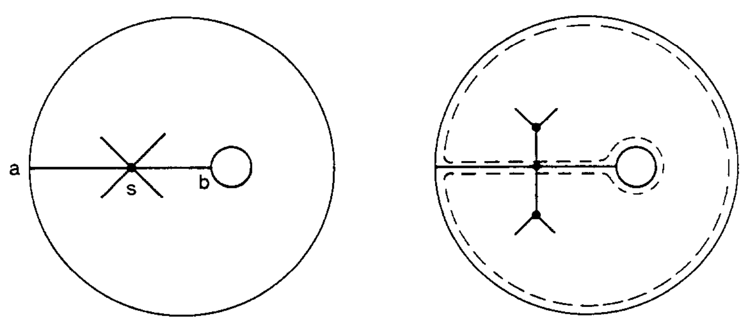

Fig. 11.

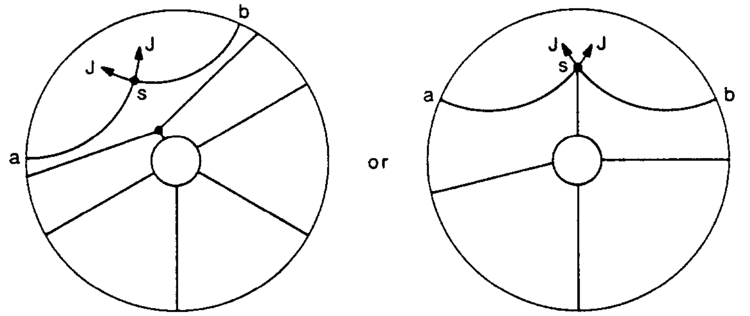

Fig. 12.

there is a point $m \in \delta A$ whose leaf goes to a point $n \in J$. Apply the extension lemma to the segment (m, n) in order to obtain the desired saddle (keeping in mind that no thorn is joined to δA).

Now we distinguish two cases.

(1) *There is a saddle s with some separatrix* (s, m) *going to* δA *and both adjacent separatrices going to* J.

There are five subcases a–e, according to how these separatrices reach J. For clarity, J and (s, m) will be in the same position on all pictures. The two points where the separatrices reach J are called A and B.

First consider subcase a (see Fig. 13). We show how to construct C on Figs 13 and 14. The part of C outside a neighborhood of J is pictured on Fig. 13 (dotted line); the part of C near J is pictured on Fig. 14. Note that for Euler characteristic reasons there has to be an

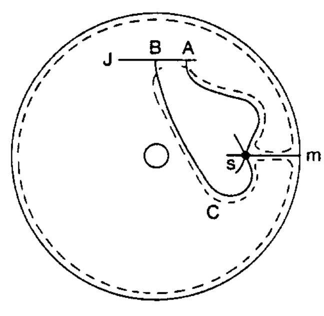

Fig. 13. Case (a).

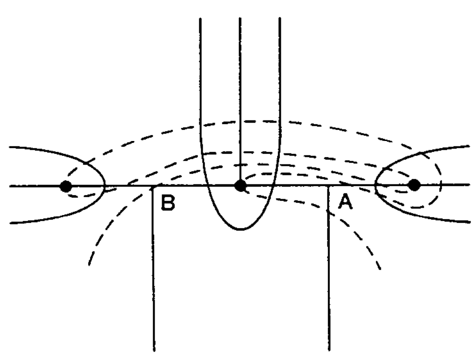

Fig. 14.

upward thorn between A and B. At other thorns (not represented on Fig. 14), C passes as in Fig. 15.

Cases b and c are dealt with similarly (see Figs 16 and 17).

Now we consider case d, when the separatrices reach J on opposite sides; there are essentially two possibilities, which we represent on Fig. 18.

The curve C is constructed outside of a neighborhood of J as before (see Fig. 18). To construct it near J, note that the separatrix of the thorn B' meets J between B and B'; hence there has to be a thorn t between A and B' or a downward thorn t' between A and B. This thorn t or t' is used as a turnaround point for C (see Fig. 19).

The remaining case, e, is pictured in Fig. 20. First assume that (s, m) is the only separatrix of s that goes to δA. Then there are two adjacent separatrices of s going to J in homotopically different ways (see Fig. 21), and it is easy to construct C in a neighborhood of J union these separatrices.

Finally, if there is another separatrix (s, n) going to δA, we recall that there is another saddle s' which is joined to δA in $A - J$ (because J is disjoint from (s_1, m_1) and (s_2, m_2)). Only

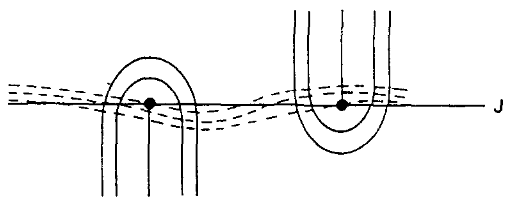

Fig. 15.

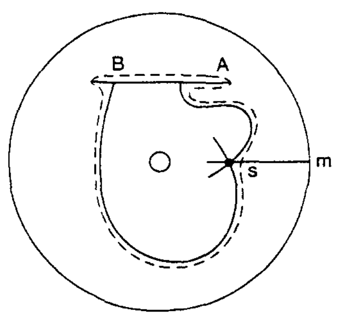

Fig. 16. Case (b).

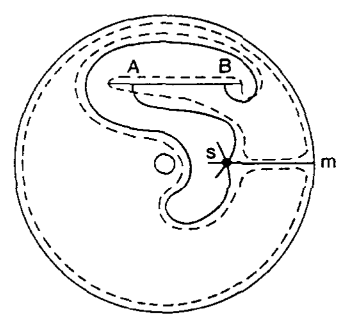

Fig. 17. Case (c).

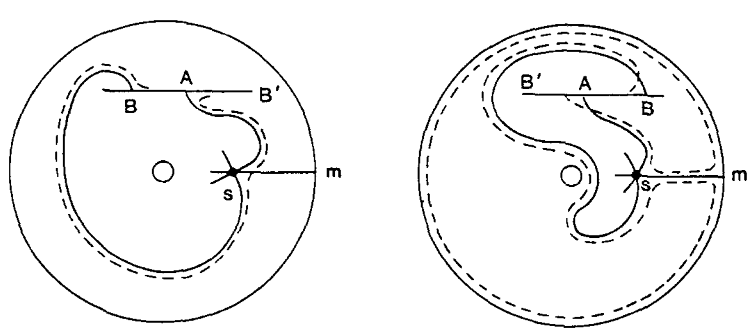

Fig. 18. Case (d).

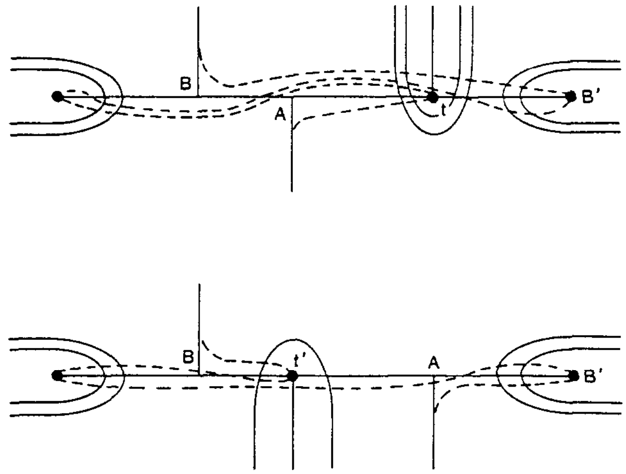

Fig. 19

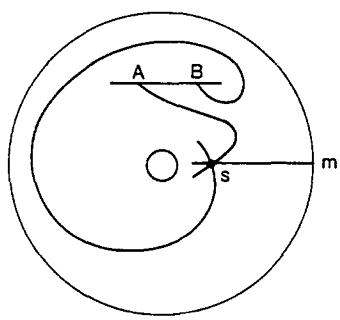

Fig. 20. Case (e).

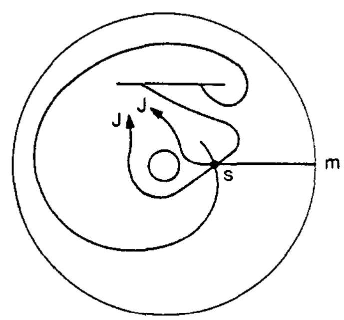

Fig. 21.

one separatrix of s' can reach δA (in $A - J$), and for the adjacent separatrices we are reduced to one of cases a, c or d.

(2) *For every separatrix going to δA, at least one of the two adjacent separatrices also goes to δA.*

Recall that there is a saddle s with a separatrix going to J and an adjacent separatrix going to δA, and that there are at least two saddles joined to δA in $A - J$. This easily implies that we are in one of two situations represented in Fig. 22 (s and s' may be tripods).

The picture shows what to do in the first situation. In the second, the separatrices (s, n) and (s', n') meet J, and we can apply the same analysis as in case 1. Subcases a and c cannot occur. In subcases b and d, the construction of C is as above. In subcase e, one has to use the separatrix of s' situated the furthest to the left (see Fig. 23); this separatrix is (s', n') if s' is a tripod.

To complete the proof of Theorem 4, we now have to show that we had the right to assume J disjoint from (s_1, m_1) and (s_2, m_2). In order to do that, we can apply the arguments used above (first step of proof of Theorem 2), provided we know that *any leaf different from (s_1, m_1) and (s_2, m_2) meets some (immersed) closed transverse curve disjoint from (s_1, m_1) and (s_2, m_2)*. We shall now prove this fact, under the simplifying assumption that no regular leaf goes from δA to δA. We leave the general case to the reader.

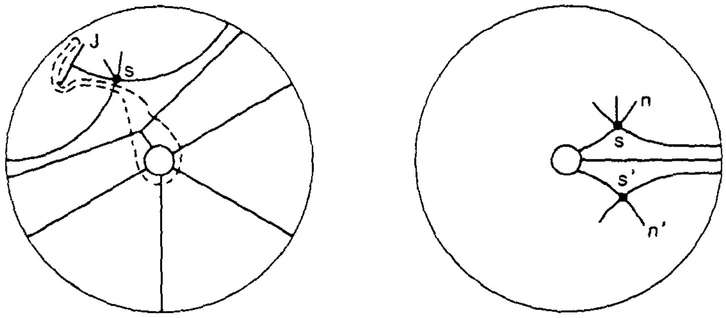

Fig. 22.

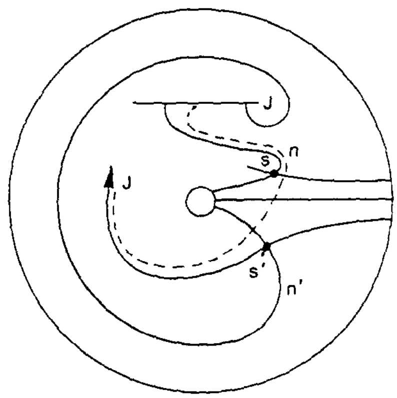

Fig. 23.

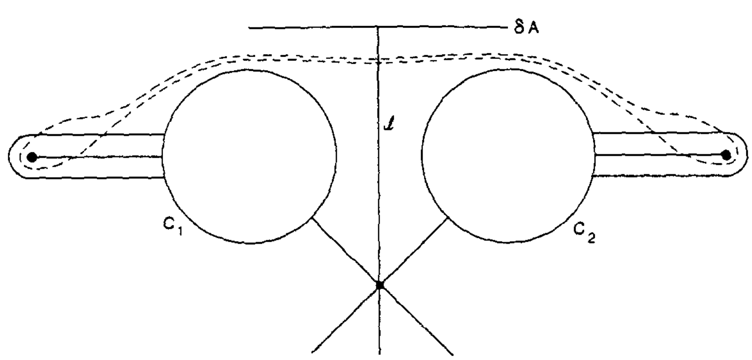

Fig. 24.

Consider in $N = M - [\text{Sing}\,\mathscr{F} \cup (s_1, m_1) \cup (s_2, m_2)]$ the union W of all leaves meeting some immersed transverse curve C with the following two properties: C is disjoint from (s_1, m_1) and (s_2, m_2), and C meets at least one thorn separatrix. We will show that $W = N$.

The set W is nonempty, open, and its frontier in N consists of separatrices of finite length (since a leaf in the frontier cannot meet a transverse curve contained in N). Note that each component of $M - \text{Sing}\,\mathscr{F}$ – (separatrices of finite length) contains at least one thorn (this is where we use our simplifying assumption) and therefore is contained in W. It follows that a separatrix l in the frontier of W is adherent to W on both sides, hence contained in W (see Fig. 24). □

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UER de Mathématiques,
Université Paris 7,
2 place Jussieu,
75251 Paris Cedex 05,
France.