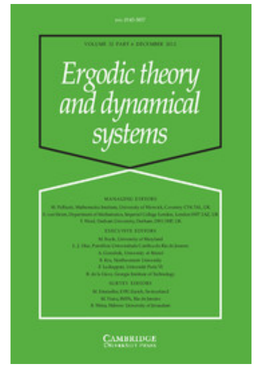


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# On the cost of generating an equivalence relation

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*Abstract.* Given a measure-preserving equivalence relation $R$ with countable classes, we study relations between the properties of $R$ and metric invariants. We give applications to pseudogroups of measure-preserving homeomorphisms.

## 0. *Introduction and statement of results*

Let $R$ be a measurable equivalence relation with countable equivalence classes on a standard Borel space $X$. We assume that there is an $R$-invariant probability measure $\mu$ and we want to relate the properties of $R$ to metric invariants, especially what we call the cost of generating $R$ (as measured with respect to $\mu$).

Our motivation and interest come mainly from topology and dynamics: $X$ is a compact metric space and classes of $R$ are orbits of a measure-preserving pseudogroup of homeomorphisms (a special example is provided by pseudogroups of isometries on the circle or on the line, see **[Le2]**, **[GLP1]**, **[GLP2]**, **[Ga]**).

Our main results, however, have nothing to do with topology, so that we first work in a purely measure-theoretic context.

To establish the ideas, we may view the classes of $R$ as orbits of an action of a countable group $G$ by $\mu$-preserving automorphisms **[FM]**. If $G$ is finitely generated, with $k$ generators, we think of $R$ as generated by $k$ automorphisms defined on the whole of $X$, for a total cost of $k$.

In general there are cheaper ways to generate $R$, using partially defined isomorphisms.

Let a *generating system* for $R$ be a countable family $\Phi = (\varphi_j : A_j \to B_j)_{j\in J}$ of $\mu$-preserving isomorphisms between Borel subsets of $X$, such that $R$ is the smallest equivalence relation satisfying $\varphi_j(x) \sim x$ for all $j \in J$ and $x \in A_j$. We often refer to the equivalence class $R(x)$ of $x$ as the *orbit* of $x$ under $\Phi$.

The *cost* of $\Phi$ is

$$\ell(\Phi) = \sum_{j\in J} \mu(A_j) = \frac{1}{2}\int_X a(x)\, d\mu(x),$$

where $a(x) \in \mathbb{N} \cup \{\infty\}$ is the number of sets $A_j$, $B_j$ containing $x$. The number $\ell(\Phi)$ is nonnegative, possibly infinite. For instance, if $R$ is amenable, it admits a generating system $\Phi$ consisting of only one element [**CFW**] and $\ell(\Phi) \leq 1$.

We shall relate $\ell$ to a number $e(R)$, which should be viewed as the 'measure of the quotient space $X/R$' (see [**Le2**]). The following proposition may be compared with Rohlin's lemma in ergodic theory.

PROPOSITION 1. *Let $R$ be a measure-preserving equivalence relation with countable classes on the standard probability space $(X, \mu)$. The following three numbers are equal:*
(i) $e_1(R) = \int_X \frac{1}{n_x}\, d\mu(x)$, *where $n_x$ is the cardinality of the class $R(x)$ and $\frac{1}{\infty} = 0$.*
(ii) $e_2(R) = \inf\{\mu(A) \mid A$ meets every class at least once$\}$.
(iii) $e_3(R) = \sup\{\mu(A) \mid A$ meets every class at most once$\}$.

The common value of these numbers will be denoted $e(R)$, or $e(\Phi)$ if $\Phi$ is a generating system for $R$. Note that $e(R) = 0$ if and only if almost every class is infinite.

THEOREM 2. *Let $\Phi$ be a generating system for the equivalence relation $R$. Then $e(\Phi) + \ell(\Phi) \geq 1$ always holds. The number $e(\Phi) + \ell(\Phi)$ is equal to 1 if and only if $R$ is amenable and the generators $\varphi_j$ are independent.*

The generators are *independent* if there is no relation, a *relation* being defined as a nontrivial reduced word $w$ in the letters $\varphi_j$ and $\varphi_j^{-1}$ whose fixed point set has positive measure (in particular, the domain of $w$ must have positive measure). A finite set of independent generators defines a treeing of $R$ in the sense of [**Ad**]; conversely, a treed equivalence relation admits independent generators (Proposition 7).

A special case of Theorem 2 is proved in [**Le2**, Corollaire II.5]. Being able to assert $\ell(\Phi) = 1$ was important in [**GLP1**] (to detect interval exchange transformations) and in [**GL**] (to compute the exact dimension of the boundary of Culler–Vogtmann's outer space).

Theorem 2 implies for instance that one has to pay a minimum price of 1 in order to make every class infinite; the price is higher if one wants to create a nonamenable relation.

Given $R$, a natural question is to determine $\ell(R)$, defined as the infimum of $\ell(\Phi)$ over all generating systems. For instance, is $\ell(R)$ strictly bigger than 1 if $R$ comes from a free action of a nonabelian free group? If so, this would lead to a nontrivial numerical invariant for (nonamenable) discrete groups.

The proof of Theorem 2 uses a theorem by Adams [**Ad**]. Consider an equivalence relation with a treeing. This means that almost every equivalence class $R(x)$ is (the set of vertices of) a locally finite tree, in a measurable way. One may then study such properties as growth or number of ends. Adams proved that $R$ is amenable if and only if $R(x)$ has at most 2 ends for almost every $x$ (for our purposes, this may be taken as a definition of amenability).

Adams also proved that almost every class with $\geq 3$ ends has exponential growth. Denoting $B_n(x)$ the number of points in $R(x)$ at distance $\leq n$ from $x$, we prove:

PROPOSITION 3. *Let $R$ be a treed equivalence relation. Almost every equivalence class* *with 2 ends has linear growth: the set of $x \in X$ such that $R(x)$ has 2 ends and $\limsup_{n\to\infty} B_n(x)/n = +\infty$ has measure 0.*

Conversely, an argument by Gaboriau implies that almost every class has 0 or 2 ends if growth is uniformly linear (see Proposition 8).

We now turn towards topology. We suppose that $X$ is a compact connected metric space and that $\mu$ is positive on every nonempty open set. We consider a countably generated pseudogroup $\Gamma$ of $\mu$-preserving homeomorphisms between open subsets of $X$ (we shall recall the definition of a pseudogroup in part III). Elements in a generating system $\Phi$ are required to have open domains. Strict inequalities may then be forced by topology rather than by measure theory.

One can associate with (the equivalence class of) $\Gamma$ a group $\pi_1 B\Gamma$, the fundamental group of Haefliger's classifying space $B\Gamma$ (see **[Ha]**, **[Sa]**). There is a natural homomorphism $\rho : \pi_1 X \to \pi_1 B\Gamma$, and we define $\overline{\pi_1 B\Gamma}$ as the quotient of $\pi_1 B\Gamma$ by the normal subgroup generated by the image of $\rho$.

PROPOSITION 4. *Let $\Phi$ be a generating system for $\Gamma$. If $\overline{\pi_1 B\Gamma}$ is not free, or if it is free of rank strictly lower than the cardinality of $\Phi$, then $e(\Phi) + \ell(\Phi) > 1$.*

If a group $G$ acts on $X$ by homeomorphisms, restrictions to open sets of elements of $G$ generate a pseudogroup. We call such pseudogroups *homogeneous*.

PROPOSITION 5. *Let $\Gamma$ be the homogeneous pseudogroup generated by an action of a countable group $G$ by $\mu$-preserving homeomorphisms. If $\Phi$ is a generating system for $\Gamma$, then $e(\Phi) + \ell(\Phi) > 1$ unless $\Phi$ consists of a single element and the action of $G$ factors through $\mathbb{Z}$.*

Our last result is a generalization of Lemma III.5 of **[Le1]**. It implies that certain pseudogroups of isometries are homogeneous, or are limits of homogeneous pseudogroups.

Let $G$ be a group acting by isometries on a metric space $X$. Let $\Phi$ be a generating system for a pseudogroup $\Gamma$ such that each $\varphi_j$ is the restriction to $A_j$ of the action of an element $g_j \in G$. For $\varepsilon > 0$ we let $\Gamma_\varepsilon$ be the pseudogroup generated by the restriction of $g_j$ to the $\varepsilon$-neighborhood of $A_j$.

PROPOSITION 6. *Let $G$ be a group acting by isometries on a connected metric space $X$ (possibly noncompact). Let $\Gamma$ be a pseudogroup generated by restrictions of elements $g_j \in G$ to open subsets $A_j$. Assume that $G$ is abelian and some orbit of $\Gamma$ is dense in $X$. Then for $\varepsilon > 0$ the pseudogroup $\Gamma_\varepsilon$ is homogeneous and independent of $\varepsilon$.*

## 1. *Proof of Proposition 1 and Theorem 2*

First we prove Proposition 1. It suffices to do so under the hypothesis that $n_x$ is constant and bigger than 1. Let $q \in \mathbb{N}$ be the value of $n_x$ if it is finite, a large integer otherwise.

By **[FM]** we may assume that classes of $R$ are orbits of a $\mu$-preserving action of a countable group $G$.

LEMMA. *There exists a countable family of (possibly empty) subsets $A_i \subset X$ such that:*
(i) *every orbit of $G$ meets exactly one $A_i$;*
(ii) *for each $i$, there exist $q-1$ elements $g_1^i, \dots, g_{q-1}^i \in G$ such that the sets $A_i, g_1^i A_i, \dots, g_{q-1}^i A_i$ are pairwise disjoint.*

*Proof.* Make a list $\kappa_1, \kappa_2, \dots$ of all $(q-1)$-uples of distinct elements of $G$. Let $\kappa_1 = (g_1^1, \dots, g_{q-1}^1)$ be the first in the list. To define $A_1$, consider sets $A \subset X$ such that $A, g_1^i A, \dots, g_{q-1}^i A$ are pairwise disjoint. If the empty set is the only such set, we let $A_1 = \emptyset$. Otherwise, a standard argument based on Zorn's lemma shows that there exists such a set which is maximal (up to sets of measure 0). We call it $A_1$.

In general, we consider the $i^{\text{th}}$ element $(g_1^i, \dots, g_{q-1}^i)$ in the list. We let $A_i$ be maximal among sets $A$ such that $A, g_1^i A, \dots, g_{q-1}^i A$ are pairwise disjoint and no $G$-orbit meets both $A$ and $A_1 \cup \dots \cup A_{i-1}$.

The only nontrivial thing to prove is that (almost) every orbit meets some $A_i$. By way of contradiction, assume that there is a $G$-invariant set $B$ of positive measure disjoint from every $A_i$. Since orbits have $q$ elements or are infinite, there exists an index $i$ such that the $q$ points $x, g_1^i x, \dots, g_{q-1}^i x$ are distinct, for $x$ in a subset of $B$ of positive measure. By standardness of the Borel space $X$, we can find $C \subset B$ of positive measure with $C, g_1^i C, \dots, g_{q-1}^i C$ disjoint. This contradicts the maximality of $A_i$. □

If $G$-orbits have cardinality $q$, we note that $A = \bigcup_i A_i$ has measure $1/q$ and meets every orbit exactly once. Furthermore there exist $\mu$-preserving bijections $\varphi_j$ $(j = 1, \dots, q-1)$ from $A$ to $\bigcup_i g_j^i A_i$, defined by $\varphi_j(x) = g_j^i x$ if $x \in A_i$. Given any $Y \subset X$, let $Y_A$ be the set of points of $A$ whose orbit meets $Y$. If $Y$ meets every orbit at most once, then

$$\mu(Y) = \mu(Y_A) \leq \mu(A) = 1/q.$$

If $Y$ meets every orbit at least once, then $Y_A = A$ and $\mu(Y) \geq \mu(Y_A) = 1/q$. This proves Proposition 1 if orbits are finite.

If orbits are infinite, then $A$ has measure $\leq 1/q$ and meets every orbit at least once. Since $q$ is arbitrary, we have proved $e_1(R) = e_2(R)$. Now suppose $Y$ meets every orbit at most once. We want to prove $\mu(Y) = 0$. Define a $\mu$-preserving injection $i : Y \to A$ as follows: if $g_1, \dots, g_n, \dots$ is an ordering of the elements of $G$, then $i(y) = g_n y$ where $n$ is the smallest index such that $g_n y \in A$. We get $\mu(Y) \leq 1/q$ for all $q$, so that $\mu(Y) = 0$. □

*Remark.* The proof shows that the supremum in the definition of $e_3$ is always achieved. The infimum in the definition of $e_2$ is achieved if and only if almost every class is finite. □

Before proving Theorem 2, we note:

PROPOSITION 7. *An equivalence relation $R$ defined by a system of independent generators with $\ell(\Phi) < +\infty$ is treeable (in the sense of [Ad]). Conversely, a treeable equivalence relation may be defined by a system of independent generators.*

*Proof.* If generators are independent, we make almost every equivalence class into a tree by placing an edge between $x$ and $y$ if and only if there exists $j \in J$ such that

$y = \varphi_j^{\pm 1}(x)$. The assumption $\ell(\Phi) < +\infty$ guarantees that almost every tree is locally finite.

Conversely, suppose $R$ is treed. View equivalence classes as orbits of an action of $G$ as before. Make a list $g_1, \dots, g_n, \dots$ of elements of $G$. We construct a system of independent generators $(\varphi_n)$, where $\varphi_n$ is the restriction of the action of $g_n$ to the set of $x$ such that there is an edge between $x$ and $g_n x$ and there is no $m < n$ with $g_n(x) = g_m^{\pm 1}(x)$ (to avoid unpleasant technicalities, we may assume that $G$ has no element of order 2).

□

*Proof of Theorem 2.*

• Our first goal is to prove the inequality $e(\Phi) + \ell(\Phi) \geq 1$.

First let $\Psi$ be any family of $\mu$-preserving bijections. Let $\Psi'$ be obtained by adding to $\Psi$ a new generator $\gamma : U \to V$. Then

$$\begin{aligned} e(\Psi') &\leq e(\Psi) \\ \ell(\Psi') &\geq \ell(\Psi) \\ e(\Psi') + 2\ell(\Psi') &\geq e(\Psi) + 2\ell(\Psi). \end{aligned}$$

The first two inequalities are obvious. The third one holds since, given any set $A$ meeting every orbit of $\Psi'$, the set $A \cup U \cup V$ meets every orbit of $\Psi$.

We shall now show that there exists a set $B$ meeting every orbit of $\Psi$, with $\mu(B) \leq \mu(A) + \mu(U)$. This will yield the inequality

$$e(\Psi') + \ell(\Psi') \geq e(\Psi) + \ell(\Psi). \qquad (*)$$

Any point $x \in X$ may be sent into $A$ by successively applying $\gamma, \gamma^{-1}$, and elements of $\Psi$. Let $s(x) \in \mathbb{N}$ be the minimum number of times one has to apply either $\gamma$ or $\gamma^{-1}$ (the minimum being taken over all possible ways to send $x$ into $A$). Define:

$$B_1 = \{x \in U \mid s(x) > s(\gamma x)\}$$

$$B_2 = \{x \in V \mid s(x) > s(\gamma^{-1} x)\}.$$

Then $\mu(B_1) + \mu(B_2) \leq \mu(U)$ since $B_1$ and $\gamma^{-1} B_2$ are disjoint subsets of $U$. On the other hand $B = A \cup B_1 \cup B_2$ clearly meets every orbit of $\Psi$. We have thus proved $(*)$.

We can now prove $e(\Phi) + \ell(\Phi) \geq 1$ for any system $\Phi = (\varphi_j)_{j \in J}$. If $J$ is finite we argue by induction on its cardinality, using $(*)$ and noting that $e = 1$ and $\ell = 0$ for the empty system. If $J$ is infinite, we may assume that $J = \mathbb{N}$ and that $\ell(\Phi)$ is finite. Let $\Phi_n = (\varphi_j)_{j=1,\dots,n}$. Writing

$$e(\Phi) + 2\ell(\Phi) \geq e(\Phi_n) + 2\ell(\Phi_n) \geq 1 + \ell(\Phi_n),$$

we get $e(\Phi) + \ell(\Phi) \geq 1$ because $\ell(\Phi) - \ell(\Phi_n)$ goes to 0 as $n$ goes to $\infty$ (since $e(\Phi) \leq e(\Phi_n)$, we see that in fact $e(\Phi) + \ell(\Phi)$ is the limit of $e(\Phi_n) + \ell(\Phi_n)$).

• Now we prove $e(\Phi) + \ell(\Phi) > 1$ if there is a relation. Suppose some nontrivial element $\varphi_{j_1}^{\varepsilon_1} \dots \varphi_{j_p}^{\varepsilon_p}$ (with $\varepsilon_i = \pm 1$) is the identity on a set $A$ of positive measure. We may assume that $p$ has been taken to be minimal. The points $x, \varphi_{j_p}^{\varepsilon_p}(x), \varphi_{j_{p-1}}^{\varepsilon_{p-1}} \varphi_{j_p}^{\varepsilon_p}(x), \dots, \varphi_{j_2}^{\varepsilon_2} \dots \varphi_{j_p}^{\varepsilon_p}(x)$ are then distinct for almost every

$x \in A$. By standardness of $X$ there exists $B \subset A$ of positive measure with $B, \varphi_{j_p}^{\varepsilon_p}(B), \ldots, \varphi_{j_2}^{\varepsilon_2} \ldots \varphi_{j_p}^{\varepsilon_p}(B)$ disjoint. Replacing the generator $\varphi_{j_p}$ by its restriction to $A_{j_p} \setminus B$ (resp. $A_{j_p} \setminus \varphi_{j_p}(B)$) gives a system $\Phi'$ with the same orbits. We then have:

$$e(\Phi) + \ell(\Phi) = e(\Phi') + \ell(\Phi') + \mu(B) \geq 1 + \mu(B) > 1.$$

• We may now assume that the generators $\varphi_j$ are independent. The important case is when $\ell(\Phi)$ is finite. The equivalence relation $R$ is then treed (see Proposition 7). Adams has proved [**Ad**] that $R$ is amenable if and only if almost every equivalence class, viewed as a tree, has at most 2 ends. We shall now complete the proof (when $\ell(\Phi)$ is finite) by showing that

$$e(\Phi) + \ell(\Phi) = 1 \text{ if and only if almost every class has at most } 2 \text{ ends.}$$

First we treat a simple case: assume that almost every $x$ belongs to at least two of the sets $A_j$, $B_j$ (equivalently: almost every class is a tree in which every vertex has valence $\geq 2$). Then $e(\Phi) + \ell(\Phi) = 1$ if and only if almost every $x$ belongs to *exactly* two of the sets $A_j$, $B_j$. This is equivalent to saying that almost every tree is isomorphic to the line, and, in particular, has two ends.

In the general case we use a process of *erasing*. Such a process was successfully used by Rips in his study of actions on $\mathbb{R}$-trees (see [**GLP1**]). We shall presently explain a way of replacing a system $\Phi$ by a new system $\Phi'$. Since $\Phi'$ will be defined on a subset $X'$ of $X$, we shall not compare $e + \ell$ to 1 but to the total measure of the space, to be denoted $m$.

Given $\Phi$, let $I$ be the set of points $x \in X$ that do not belong to any of the sets $A_j$, $B_j$. Let $U_j$ be the set of points belonging to $A_j$, but not to $B_j$ or any other set $A_k$, $B_k$ $(k \neq j)$. Similarly, let $V_j$ be the set of points belonging to $B_j$ but not to $A_j$ or any $A_k$, $B_k$. Let $X'$ be obtained from $X$ by removing $I$ and the union of all sets $U_j$, $V_j$. We define a system $\Phi'$ on $X'$ by restricting each $\varphi_j$ to $A_j \setminus (U_j \cup \varphi_j^{-1} V_j)$.

The effect of the modification on the orbits of $\Phi$, viewed as trees, is the following. Orbits consisting of one or two points disappear (they do not meet $X'$). Other orbits lose their vertices of valence 1, together with the corresponding open edge. In particular, the number of ends does not change.

The difference $e+\ell-m$ is the same for $\Phi$ and $\Phi'$, as shown by the following formulas ($t$ denotes the measure of the union of all $\Phi$-orbits consisting of 2 elements):

$$\begin{aligned} e(\Phi') &= e(\Phi) - \mu(I) - t \\ \ell(\Phi') &= \ell(\Phi) - \sum_j \mu(U_j) - \sum_j \mu(V_j) + t \\ m(\Phi') &= m(\Phi) - \mu(I) - \sum_j \mu(U_j) - \sum_j \mu(V_j). \end{aligned}$$

Having defined the elementary process $\Phi \mapsto \Phi'$, we can now define a sequence $\Phi^n$, with $\Phi^0 = \Phi$ and $\Phi^{n+1} = (\Phi^n)'$. Each $\Phi^n$ is defined on a space $X^n$ and has generators $\varphi_j^n$ defined on smaller and smaller sets $A_j^n$. We consider the limiting system $\Phi^\infty$ defined on $X^\infty = \bigcap_n X^n$, with generators $\varphi_j^\infty$ defined on $\bigcap_n A_j^n$. The theorem holds for $\Phi^\infty$ because we are in the special case treated above.

The equality $e + \ell = m$ holds for $\Phi^\infty$ if and only if it holds for $\Phi$. On the other hand, $\Phi$-orbits with less than 2 ends are completely erased: they do not meet $X^\infty$. The orbits of $\Phi^\infty$ are precisely the intersections with $X^\infty$ of $\Phi$-orbits with at least 2 ends, and the number of ends is preserved. It follows that almost every $\Phi$-orbit has at most 2 ends if and only if almost every $\Phi^\infty$-orbit has at most 2 ends, and the theorem is proved for $\ell(\Phi)$ finite.

• Finally we note that, if $\ell(\Phi)$ is infinite, the equivalence relations generated by the finite systems $\Phi_n$ used in the proof of $e + \ell \geq 1$ are not amenable. It follows that $R$ is not amenable. □

2. *Proof of Proposition 3*

Let $T$ be a locally finite tree with 2 ends. It has a trunk $T^\infty$, the unique geodesic joining the ends. Every component of $T \setminus T^\infty$ is a finite tree. There is a projection $\pi : T \to T^\infty$ sending a point $y$ to the point in $T^\infty$ closest to $y$. For $x \in T^\infty$ we let $g(x)$ be the cardinality of the preimage $\pi^{-1}(x)$.

Let $B_n(x)$ be the number of points of $T$ at distance $\leq n$ from $x$. If $x \in T^\infty$ we have

$$B_n(x) \leq \sum_y g(y), \qquad (**)$$

the sum being taken over the $2n + 1$ points in $T^\infty$ at distance $\leq n$ from $x$.

Now consider a treed relation $R$. In order to prove Proposition 3 we may assume that almost every class has 2 ends. Recall the erasing process used in the proof of Theorem 2. The 'limit set' $X^\infty$ is simply the union of all trunks (up to a set of measure 0). Note that it has positive measure since classes are countable. We consider the function $g$ defined on $X^\infty$ as above. It belongs to $L^1(X^\infty)$ since

$$\int_{X^\infty} g(x)\, d\mu(x) = \mu(X) = 1.$$

Assume for the moment that we can select one of the two ends of the tree $R(x)$ in a measurable way. We then get an automorphism $f$ of $X^\infty$ as follows: a point $x \in X^\infty$ belongs to the trunk of $R(x)$ and we let $f(x)$ be its closest neighbor in the trunk, in the positive direction. Using $(**)$ we get:

$$\frac{B_n(x)}{n} \leq \frac{1}{n} \sum_{i=-n}^{n} g(f^i(x))$$

for $x \in X^\infty$. Since $g \in L^1(X^\infty)$, the right-hand side has a finite limit for almost every $x \in X^\infty$ when $n \to +\infty$, by Birkhoff's ergodic theorem. We thus get the finiteness of $\limsup_{n \to +\infty} \frac{B_n(x)}{n}$ for almost every $x \in X^\infty$, hence also for almost every $x \in X$.

There remains the problem of orienting the trunks. This is taken care of by 'passing to a 2-sheeted covering' (cf. [**Gh**]). □

The following partial converse, based on [**Ga**, Proposition VI.1], was worked out with Gaboriau.

PROPOSITION 8. *Let $R$ be a treed equivalence relation. Assume there exists a constant $C$ such that $B_n(x) \leq Cn$ for every $n \geq 1$ and almost every $x \in X$. Then almost every class has 0 or 2 ends.*

*Proof.* Since classes with at least three ends have exponential growth by [**Ad**], we assume that almost every class has exactly one end and we argue towards a contradiction.

Once again we apply the erasing process. Consider the set $X^p$ consisting of points that remain after $p$ steps. Since classes have one end, the limit set $X^\infty = \bigcap_p X^p$ has measure 0 and $\mu(X^p)$ goes to 0 as $p$ goes to infinity.

Fix integers $p < n$. For $x \in X$ let $B_n^p(x)$ be the number of points of $R(x)$ that are at distance $\leq n$ from $x$ and belong to $X^p$. Note the inequality $B_n^p(x) \geq n - p$: on the infinite ray joining $x$ to the end of $R(x)$, at most $p$ points may lie outside of $X^p$. We then write

$$n - p \leq \int_X B_n^p(x)\, d\mu(x) = \int_{X^p} B_n(x)\, d\mu(x) \leq Cn\mu(X^p).$$

Letting $n$ go to infinity we find that $\mu(X^p)$ is bounded away from 0, a contradiction. □

3. *Proof of Propositions 4, 5 and 6*

First recall that a pseudogroup of homeomorphisms of a space $X$ is a collection $\Gamma$ of homeomorphisms $\gamma : U \to V$ between open subsets of $X$ satisfying the following conditions:

(i) The identity map of $X$ belongs to $\Gamma$.
(ii) If $\gamma \in \Gamma$ and $U' \subset U$ is open, then the restriction of $\gamma$ to $U'$ belongs to $\Gamma$.
(iii) If $\gamma \in \Gamma$, then $\gamma^{-1} : V \to U$ belongs to $\Gamma$.
(iv) If $\gamma, \gamma' \in \Gamma$, then the composition $\gamma' \circ \gamma$ (defined on $\gamma(U) \cap U'$) belongs to $\Gamma$.
(v) If every $x \in U$ has a neighborhood $W$ such that the restriction of $\gamma$ to $W$ belongs to $\Gamma$, then $\gamma \in \Gamma$.

Given a set of homeomorphisms $\gamma_j : U_j \to V_j$ (such as those obtained from a group action), the pseudogroup *generated* by this set is the smallest pseudogroup containing each $\gamma_j$.

*Proof of Proposition 4.* We may assume that the domain of each $\varphi_j$ is connected. The group $\overline{\pi_1 B\Gamma}$ then admits the following presentation: generators are elements of $J$, relators are words $j_1^{\varepsilon_1} \dots j_p^{\varepsilon_p}$ such that $\varphi_{j_1}^{\varepsilon_1} \dots \varphi_{j_p}^{\varepsilon_p}$ is the identity on some nonempty open set. The hypothesis on $\overline{\pi_1 B\Gamma}$ guarantees that there is a nontrivial relator. Since $\mu$ has full support, the generators $\varphi_j$ are not independent. The result follows. □

*Proof of Proposition 5.* Suppose $e(\Phi) + \ell(\Phi) = 1$. Then the generators of $\Phi$ are independent. Let $g$ be an element of $G$ that acts non-trivially. Each $x \in X$ has a neighborhood on which the action of $g$ is represented by some reduced word $w$ in the generators $\varphi_j^{\pm 1}$. This $w$ is uniquely determined, since generators are independent. It follows that $w$ has to be defined on the whole of $X$ (recall that $X$ is connected). Consequently $\Phi$ consists of only one element $\varphi$, since otherwise $\ell(\Phi)$ would be bigger than one. The action of any $g \in G$ is given by some power $\varphi^{\tau(g)}$, and the action of $G$ factors through the homomorphism $\tau : G \to \mathbb{Z}$. □

*Proof of Proposition 6.* First note that $\Gamma_\varepsilon$ depends on $\Gamma$, but not on the chosen set of generators $\varphi_j = g_j | A_j$. Also $(\Gamma_\varepsilon)_{\varepsilon'} = \Gamma_{\varepsilon+\varepsilon'}$.

The existence of a dense orbit for $\Gamma$ implies that every orbit of $\Gamma_t$ is dense for $t > 0$, so we may as well assume that every $\Gamma$-orbit is dense.

Fix $\varepsilon > 0$. Given $g \in G$, let $U_g$ be the largest open set such that the restriction of $g$ to $U_g$ belongs to $\Gamma_\varepsilon$. We shall show that $U_g$ *is either empty or equal to the whole of* $X$. This will imply Proposition 6: $\Gamma_\varepsilon$ is the homogeneous pseudogroup generated by the action of $H$, the subgroup of $G$ generated by the elements $g_j$ (we tacitly suppose $A_j \neq \emptyset$).

Let $h$ be any element of $G$. Assuming that $U_h$ contains some nonempty open set $O$, we first show that $d(y, hy) < \varepsilon \Rightarrow y \in U_h$ (of course $d$ denotes distance in $X$). Let $\gamma$ be an element of $\Gamma$ representing some $v \in G$ and taking $y$ into $O$: it exists because we assume that every $\Gamma$-orbit is dense. Then $hy \in U_v$ since $d(y, hy) < \varepsilon$, and $hvy = vhy$ belongs to $U_{v^{-1}}$ (recall that $G$ is abelian). It follows that $y \in U_h$: we can go from $y$ to $hy$ in $\Gamma_\varepsilon$, by applying $\gamma$, then $h$, then $v^{-1}$.

Next we show that $U_g$ is dense in the $\varepsilon$-ball around $x$ if $x \in U_g$. Let $y$ be $\varepsilon$-close to $x$. Choose elements $\gamma_n \in \Gamma$, representing $h_n \in G$, such that $\gamma_n x$ converges to $y$. Then $d(gx, h_n gx) = d(gx, gh_n x) = d(x, h_n x)$ is $< \varepsilon$ for $n$ large, and $gx$ belongs to $U_{h_n}$ by the preceding argument (applied to $h_n$). It follows that $\gamma_n x$ belongs to $U_g$.

Since $X$ is connected, we have now proved that every $U_g$ is either empty or dense. Applying this to $\Gamma_{\varepsilon'}$ with $\varepsilon' < \varepsilon$, we see that $U_g$ is indeed empty or equal to $X$. □

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