Vol. 55, No. 4 DUKE MATHEMATICAL JOURNAL © December 1987

COMPLETE MINIMAL SURFACES WITH LONG LINE BOUNDARIES

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In this paper we will study complete minimal surfaces (CMS's) bounded by lines in $\mathbb{R}^3$. By complete we mean that a path that leaves every compact subset of a manifold has infinite length, even if the manifold has boundary. We will prove Bernstein-type theorems for such surfaces. For example, we prove that if M is a CMS whose interior is a graph over a square J in the (x, y) plane, and if the boundary of M is composed of the four vertical lines over the vertices of J, then M is Scherk's surface. Another theorem of this type that we prove is: Let M be a CMS whose interior is a graph over an infinite strip J in the (x, y) plane, and suppose ∂M is composed of two vertical lines over points of ∂J. Then M is part of a helicoid. We obtain other results of this nature.

Minimal surfaces bounded by lines have been studied extensively. A line L on a minimal surface in $\mathbb{R}^3$ has an intrinsic meaning: M is invariant by the symmetry of $\mathbb{R}^3$ through L (the reflection principle), hence L lifts to a geodesic in the universal conformal covering space of M, with the Poincaré metric.

In his memoir [5] Riemann found a means to construct minimal surfaces with boundary a given polygon, where the sides of the polygon could be short or long (i.e., of finite or infinite length). His technique works for polygons with up to four sides. The contemporary reader may interpret Riemann's statements of theorems as announcing unicity of CMS's with given polygonal boundaries (see Darboux [2], pp. 491–492); however, the paper of Riemann only addresses the problem of existence.

Serret found an infinite family of CMS's with boundary two lines, each example distinct from a helicoid and simply connected [7]. Riemann found such examples that are not simply connected, modelled on a punctured torus [5]. Jenkins and Serrin consider the Plateau problem over convex compact domains D in the plane where the data on ∂D is discontinuous [3]. More precisely, suppose ∂D is a polygon $A_1, B_1, A_2, B_2, \dots, A_n, B_n$ (in the order indicated) and one desires a minimal surface M that is a graph over int D and takes the values $+\infty$ on each A_i, $-\infty$ on each B_i (this implies ∂M is the set of lines over the vertices of D). They prove that M exists and is unique provided $\Sigma_i^n |A_i| = \Sigma_i^n |B_i|$, where $|A|$ means the length of A. We will prove $c(M)$ is finite.

It would be interesting to know whether their techniques work on noncompact domains D.

I. The helicoid. The helicoid M is a CMS modelled on $\mathbb{C}$ having a Weierstrass representation $g(z) = e^z$, $\omega = -iae^{-z}\,dz$, a real. M is invariant by a vertical

Received August 4, 1986.

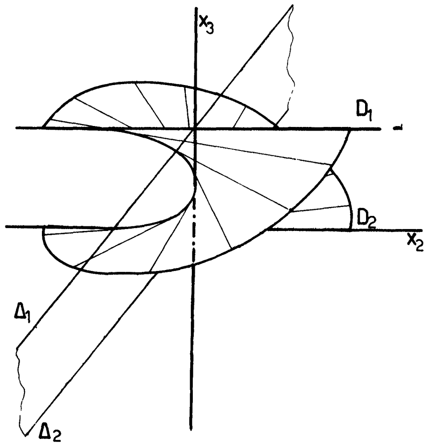

FIGURE 1

translation by a distance a and the quotient minimal surface $\hat{M}$ in $\mathbb{R}^3/Z$ is an embedded CMS of total curvature 4π, modelled on a two-punctured sphere, with a Weierstrass representation: $\hat{g}(z) = z$, $\hat{\omega}(z) = -ia\,dz/z^2$.

We take the generating lines of M to pass through the x_3 axis, and the x_2 axis to be a generating line D_2. Let D_1 be the image of D_2 by the vertical translation by a and let M_0 be the part of M bounded by $D_1 \cup D_2$. Then int M_0 is a graph over the infinite strip B in the (x_1, x_3) plane bounded by the two lines Δ_1, Δ_2, with $\Delta_i \cap D_i$ a point for $i = 1, 2$ (Figure 1).

THEOREM I.1. *Let S_0 be a CMS bounded by parallel lines L_1, L_2. Suppose* int S_0 *is a graph over the infinite strip B and L_1, L_2 are orthogonal to the plane of B; $L_i \cap \Delta_i$ a point for $i = 1, 2$. Then $S_0 = M_0$ (up to translation); in particular, the points $L_i \cap \Delta_i$ are the extremities of a segment of B, orthogonal to ∂B.*

Proof. The reflection principle for minimal surfaces bounded by lines permits us to construct a CMS S in $\mathbb{R}^3$ by reflecting S_0 through its line boundaries, and then continuing to reflect each of the surfaces obtained through their line boundaries. S is invariant by a translation of $\mathbb{R}^3$ (for which S_0, together with its reflected image through L_2, is a fundamental domain). The quotient space $\hat{S}$ of S by this translation is a CMS in $\mathbb{R}^3/Z$. We shall prove I.1 by showing that $\hat{S} =$ the helicoid $\hat{M}$.

In [6], we show that $\hat{S}$ has a Weierstrass representation $(\hat{g}, \hat{\omega})$. Clearly, the Gauss map g of S passes to the quotient, and it's easy to check that ω does as well.

Now, Osserman has proved that if N is a CMS in $\mathbb{R}^3$ that is either infinitely connected or hyperbolic, then the Gauss map of N assumes every value infinitely often, with the possible exception of a set of capacity zero [4]. We remark that this theorem applies equally well to CMS's in quotient spaces: $\mathbb{R}^3$ modulo a group of translations: the proofs in Lemma 2.1 and Theorem 2.2 of [4] work for the quotient minimal surface with the induced Weierstrass representation. This will enable us to prove $\hat{S}$ is conformally a two-punctured sphere and has finite total curvature.

Clearly, our graph hypothesis implies $\hat{S}$ is topologically a two-punctured sphere. Hence, to show $\hat{S}$ is conformally a two-punctured sphere, it suffices to show there is a set of positive capacity that is not assumed infinitely often by the Gauss map $\hat{g}$ of $\hat{S}$.

Let C be the great circle $x_1^2 + x_3^2 = 1$ on the Riemann sphere. We claim each point of C is assumed at most once by $\hat{g}$, and since the capacity of C is positive, this proves $\hat{S}$ is conformally a two-punctured sphere.

First observe that our graph hypothesis implies $g^{-1}(C) \subset L_1 \cup L_2$, so we need only show that g is injective on $L_1 \cup L_2$. Let $p \in L_1$. The tangent plane P to S at p contains L_1, and L_1 is parallel to the x_2 axis, so S intersects the plane $x_2^\perp$ at p in a curve $C(p)$. The tangent to $C(p)$ projects to a direction $C'(p)$ in B. As p traverses L_1 monotonically, the direction $C'(p)$ (based at the point $L_1 \cap B$) turns monotonically, since int S is a graph over B. Since $g(p)$ is orthogonal to $C'(p)$, this proves g is injective on L_1, as desired.

Consider a puncture of $\hat{S}$. If z were an essential singularity of $\hat{g}$, then $\hat{g}$ would assume every value infinitely often, with the possible exception of two values. Since this is not true of C, we have z a removable singularity. Thus $\hat{g}$ extends to a meromorphic map on the Riemann sphere and $\hat{S}$ has finite total curvature.

We orient $\mathbb{C}$ so that C projects to the imaginary axis by stereographic projection, and we translate M in the x_1 direction so L_1 coincides with the x_2 axis. The tangent plane to S along $L_1 \cup L_2$ is never horizontal because at a point p of $L_1 \cup L_2$ where the tangent plane T is horizontal, the graph hypothesis implies that M is on one side of T near p, so M would equal T by the boundary maximum principle. Hence, g misses 0 and ∞. We know $\hat{g}$ is meromorphic on the conformal two-point compactification (a sphere $\bar{S}$) of $\hat{S}$ and $\hat{g}^{-1}(C) \subset L_1 \cup L_2$, so $\hat{g}(L_1 \cup L_2)$ is the entire imaginary axis less the point 0 ($\hat{g}$: $\bar{S} \to = \mathbb{C} \cup \{\infty\}$ is surjective).

Let $\Sigma = \overline{L_1 \cup L_2} \subset \bar{S}$; Σ is a Jordan curve that separates $\bar{S}$ into two discs E_1, E_2. The graph condition implies that the normals to S on E_1 point into one component of $S^2 - C$, and that the normals along E_2 point into the other component. Since each component is a hemisphere, it follows that the degree of $\bar{g}$: $\bar{S} \to S^2$ is equal to the degree of $\bar{g}$: $\Sigma \to C$. This latter degree is 1. Hence we can assume $\bar{S}$ is parametrized, so that $\bar{g}(z) = iz$ and the punctures are $0, \infty$.

The form $\bar{\omega}$ has no zeros or poles in $\mathbb{C} - 0$, hence $\bar{\omega}(z) = cz^l\,dz$, $l \in Z^+$.

For notational convenience, we now omit the bars on all the data of $\bar{S}$.

We calculate the residue at 0 of the Weierstrass form Φ_3 of $\bar{S}$:

$$\begin{aligned}\operatorname{Res}(\Phi_3) &= -2\pi c \quad \text{if } l = -2\\ &= 0 \quad \text{if } l \neq -2.\end{aligned}$$

Since $\hat{S}$ is periodic, we have $\text{Re}(\text{Res}\,\Phi_3) = 2a$. Hence $l = -2$ and $\text{Re}(c) = 2a$. Since $g(z) = iz$ and g takes imaginary values only on $L_1 \cup L_2$, we have L_1 contained in the real axis. Hence $0 = x_3(L_1) = \text{Re}\int_{L_1}\Phi_3 = \text{Re}(ci\ln|t|)$, and this implies that c is real.

Thus, $\hat{S}$ has the Weierstrass representation of the helicoid, and this completes the proof of I.1.

Remark. Theorem I.1 remains true if L_1 is a boundary line of B and L_2 is a line orthogonal to the other boundary line B, for then the reflection of S through L_1 yields a surface S' satisfying the hypothesis of I.1; L_2 is reflected to a line L_1' parallel to L_2 and int S' is a graph over the strip $B \cup B'$, where B' is the strip B reflected across L_1.

The theorem is also true if L_1 and L_2 are the boundary lines of B, for then the successive reflections of S yield a minimal surface that is a graph over the entire plane, hence it is a plane by Bernstein's theorem. Similarly, if L_1 is parallel to L_2, M is a plane.

Now we will study some other configurations of L_1, L_2.

THEOREM I.2. *Let L_1, L_2 be nonintersecting lines making an angle α. Let P be the plane orthogonal to L_1 and containing the common perpendicular Δ to L_1 and L_2. Let B be the infinite strip in P bounded by the two lines orthogonal to Δ passing by the points E and F of intersection of Δ with L_1, L_2 (Figure 2). Then a CMS S_0 of finite total curvature with $\partial S_0 = L_1 \cup L_2$ and* int S_0 *a graph over B is part of a helicoid, or a plane.*

Remark. We believe the hypothesis of finite total curvature is not necessary for this theorem.

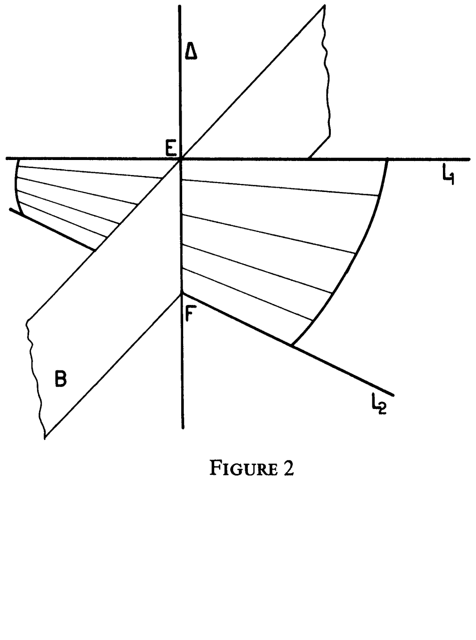

FIGURE 2

We would like to thank R. Krust and P. Collin for valuable discussions allowing us to remove the hypothesis that the angle is rational.

Proof. Let S_0 be a CMS as in I.2, and let S be the CMS obtained by reflecting S_0 through $L_1 \cup L_2$ and then successively reflecting each bounded surface obtained through each of its line boundaries. Choose the same coordinate system as in I.1, where $\alpha = \pi$; the x_1 axis parallel to $\partial\beta$, the x_2 axis parallel to L_1, and the x_3 axis parallel to the common perpendicular Δ to L_1 and L_2.

Let σ_1, σ_2 be the isometries of $\mathbb{R}^3$, which are the reflections through L_1, L_2, respectively (i.e., σ_i is rotation of $\mathbb{R}^3$ about L_i by the angle π). Let $\tau = \sigma_2\sigma_1$; τ is an orientation-preserving isometry of S. Consider the orientable surface $\hat{S} = S/\tau$. The metric on S is invariant by τ; therefore, $\hat{S}$ acquires a complete metric.

The total curvature of $\hat{S}$ is $2c(S_0)$, hence is finite. Since $\hat{S}$ is topologically a twice-punctured sphere ($S_0 \cup \sigma_1(S_0)$ is a fundamental domain for the action of τ), it follows from Huber's theorem that $\hat{S}$ is conformally $\mathbb{C}^* = \mathbb{C} - \{0\}$. Hence S is conformally $\mathbb{C}$.

For $x \in S_0$, the normal vector to S at $\sigma_1(x)$ is the image of the normal vector to S at x by inversion in the great circle of $\mathbb{S}^2$ orthogonal to L_1; here we identify a normal vector to S with its parallel translate to the origin. Hence, the normal vector to S at $\tau(x)$ is the image of the normal vector to S at x by a rotation of $\mathbb{S}^2$ about the axis Δ (which we take as the x_3 axis) by twice the angle between L_1 and L_2. Notice that S_0 has no normal vector parallel to $e_3 = (0,0,1)$. On int S_0 this follows from our graph hypothesis, and on $L_1 \cup L_2$ it follows from the boundary maximum principle. Since τ leaves the direction of e_3 invariant, it follows that the normal field to S is never parallel to e_3.

We identify $\mathbb{C}$ with $\mathbb{S}^2 - \{e_3\}$ by stereographic projection from e_3. Then the Gauss map g on S takes its values in $\mathbb{C}^*$. Let $\tilde{g}$: $S \to \mathbb{C}$ be a lifting of g to the covering space $\mathbb{C}$ of $\mathbb{C}^*$ via e^z, i.e., $g(x) = \exp(\tilde{g}(x))$. Since rotation about 0 in $\mathbb{C}$ lifts to translation in $\mathbb{C}$, we have $\tilde{g}(\tau(x)) = \tilde{g}(x) + ai$ for $x \in S$ and some real number a. We know S is conformally $\mathbb{C}$ and L_1, L_2 correspond to parallel straight lines in $\mathbb{C}$ with τ a translation. So we can conformally parametrize S by $\mathbb{C}$ so that $\tau(z) = z + ai$ for all $z \in \mathbb{C}$.

Now, $\tilde{g}(z + ai) = \tilde{g}(z) + ai$ for $z \in \mathbb{C}$, and the image by $\tilde{g}$ of the band E in $\mathbb{C}$ bounded by L_1, L_2 is contained in a horizontal band of height π (the graph

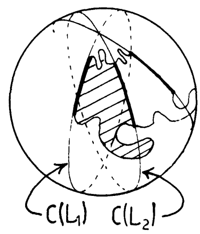

FIGURE 3

hypothesis). For $z \in \mathbb{C}$, let n be an integer satisfying $z - nai \in E$. Then $\tilde{g}(z) = \tilde{g}(z - nai) + nai$ and $\tilde{g}(z) - z = \tilde{g}(z - nai) - (z - nai)$. Hence the imaginary part of $\tilde{g}(z) - z$ is bounded on $\mathbb{C}$, so $\tilde{g}(z) - z$ is constant and $g(z) = ce^z$ for some constant c. After a conformal reparametrization, we can suppose $g(z) = e^z$.

It remains to find the ω of the Weierstrass representation. The coordinate x_3 is a harmonic function on $\mathbb{C}$ ($= S$) with no critical point, and the graph hypothesis implies that the level sets $\{x_3 = \text{constant}\}$ are connected. Let x_3^* be a harmonic conjugate of x_3. Then the restrictions of x_3^* to the level sets of x_3 have no critical points, and it follows that $\eta = x_3^* + ix_3$ is a global conformal parameter on $\mathbb{C}$. Hence we can take $\eta(z) = az$ for some $a \in \mathbb{C}$, and $g(\eta) = e^\eta$.

Now $x_3(\eta) = \operatorname{Re} \int_0^\eta g\omega = \operatorname{Re}(-\int_0^\eta i\,d\eta)$, so $\int_0^\eta g\omega$ and $-\int_0^\eta i\,d\eta$ differ by a constant and $g\omega = -i\,d\eta$. Thus $\omega = -ie^{-\eta}\,d\eta$ and S is a helicoid.

Examples. Let $P(z)$ be a complex polynomial with real coefficients, and consider the pair (g, ω):

$$g(z) = \frac{i}{P(e^z)}, \quad \omega = P(e^z)^2.$$

It is not hard to check that for $\operatorname{Im}(z) \in [0, 2\pi], (g, \omega)$ parametrizes a simply connected CMS with boundary two parallel long lines L_1, L_2. Also, the total curvature is finite and one varies the total curvature with the degree of P. Also, one varies the angle between L_1, L_2 by multiplying the data by a constant. Hence, the graph hypothesis is necessary in Theorem I.1.

II. Scherk's surface. Scherk's surface M is a CMS in $\mathbb{R}^3$ that is doubly periodic; it is invariant by translations parallel to the x_1 and x_2 axes. The surface projects onto the (x_1, x_2) plane, onto the black squares of a checkerboard pattern, and it is a graph over the interior of each black square (Figure 4). M contains the vertical lines passing through the vertices of the squares; a part of M over the interior of a square with the four vertices is a CMS bounded by the four vertical lines through the vertices. The quotient $\hat{M}$ of M by the group G ($\approx Z^2$) of translations that leave it invariant is an embedded CMS in $\mathbb{R}^3/G$, which is conformally a four-punctured sphere and of total curvature 4π. A fundamental domain for G is two black squares meeting at a vertex. A Weierstrass representation of $\hat{M}$ is

$$g(z) = z, \qquad \omega(z) = \frac{\lambda\, dz}{z^4 - 1}, \qquad \lambda \in R - (0).$$

The punctures are the four roots of unity; the limiting values of the Gauss map g at the four ends of $\hat{M}$.

We remark that Scherk's surface also exists over a rhombus tesselation of the plane.

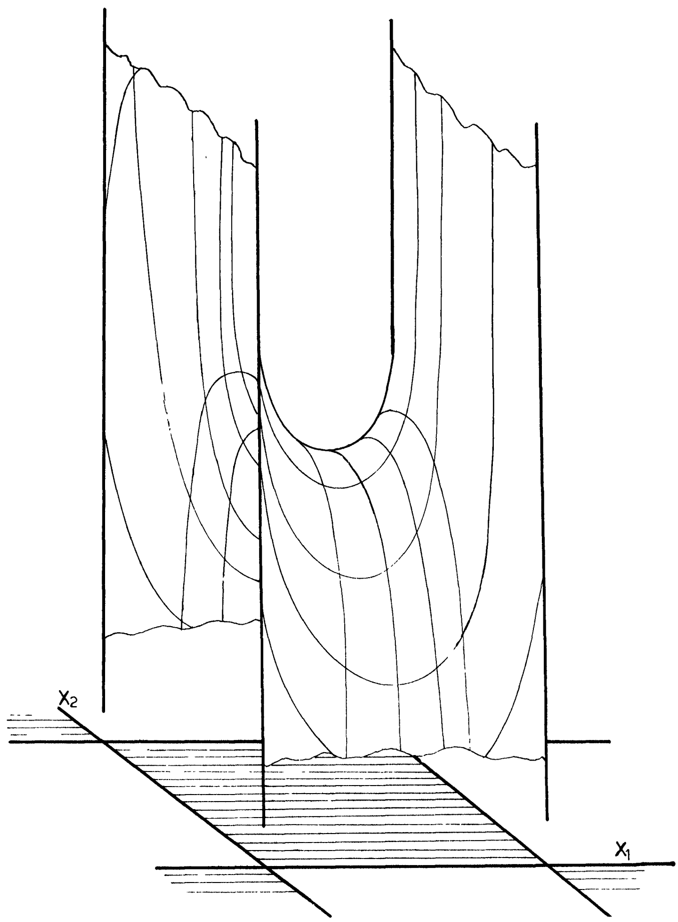

FIGURE 4

THEOREM II.2. *Let S_0 be a CMS whose interior is a graph over a parallelogram P in the (x_1, x_2) plane and whose boundary is the four vertical lines over the vertices of P. Then P is a rhombus and S_0 is Scherk's surface. A Weierstrass representation is*

$$g(z) = z, \qquad \omega(z) = \frac{\lambda\, dz}{(z - e^{i\theta})(z - e^{-i\theta})(z + e^{i\theta})(z + e^{-i\theta})},$$

$$\lambda \in R - (0).$$

Proof. Let S be the CMS on $\mathbb{R}^3$ obtained by successively reflecting S_0 across its lines boundary. S projects onto the checkerboard pattern of the plane defined by P and contains the vertical lines over the vertices. Let G be the group of horizontal translations with fundamental domain two parallelograms P, P' meeting at one vertex. Let $\hat{S}$ be the CMS S/G in $\mathbb{R}^3/G$. Notice that S is a graph over (int P) $\cup$ (int P'), and if the normal vector to S on int P points into the northern hemisphere of S^2, then the normal points into the southern hemisphere on int P'. Hence, $\hat{S}$ is an orientable surface and is topologically S^2 less four points.

The points of S with a horizontal normal vector are precisely the lines over the vertices of the checkerboard. Let g denote the Gauss map of $\hat{S}$. Exactly as in the proof of I.1, we see that g is injective on each of the four vertical lines of $\hat{S}$. Thus the equator of S^2 is finitely covered by g: $\hat{S} \to S^2$; hence, $\hat{S}$ is conformally a four-punctured sphere, each puncture is a removable singularity of g, and the total curvature of $\hat{S}$ is finite. So g extends to a meromorphic map (we also call g) on the sphere $\bar{S}$, obtained by adding the four punctures to $\hat{S}$.

Let $\bar{C}$ be the Jordan curve on $\bar{S}$ composed of the four vertical lines over the vertices of P, together with the four vertices. $\bar{C}$ separates $\bar{S}$ into two disks E_1, E_2, and if gE_1 is contained in the northern hemisphere, then gE_2 is contained in the southern hemisphere (E_1 is the part over int P and E_2 is the part over int P'). Hence, the degree of g: $\bar{S} \to S^2$ is the degree of $g/\bar{C}$: $\bar{C} \to C$, where C is the equator of S^2. We claim this degree is 1. On a vertical line L, g is injective, and the most the normal can turn as one traverses L is the angle P makes at the vertex $L \cap P$. Since g: $\bar{S} \to S^2$ is surjective and $g^{-1}(C) = \bar{C}$, it follows that g: $\bar{C} \to C$ is a bijection. Hence, the degree of g is 1 and the values of g at the punctures are $\pm e^{\pm i\theta}$ (after a rotation of P).

So we can parametrize $\bar{S}$ by its Gauss map, i.e., we take $g(z) = z$, and the punctures are $\pm e^{\pm i\theta}$. Then the form ω of the Weierstrass representation of $\bar{S}$ has no zeros in $\mathbb{C}$ and ∞ is a regular point of $\bar{S}$; hence, ∞ is a zero of ω order 2 (it's a simple pole of g). The poles of ω are at the four punctures. Hence,

$$\omega(z) = \frac{c\,dz}{(z - e^{i\theta})^{n_1}(z - e^{-i\theta})^{n_2}(z = e^{i\theta})^{n_3}(z + e^{-i\theta})^{n_4}}$$

for some $c \in \mathbb{C}$ and integers $n_k \geqslant 1$.

Since ∞ is a zero of order 2 of ω, we have $\Sigma_{k=1}^{4} n_k = 4$; hence, $n_k = 1$ for $k = 1, 2, 3, 4$.

Now we shall see that P must be a rhombus. The coordinate function x_3 is well defined on $\hat{S}$, since G leaves x_3 invariant. Hence, $\mathrm{Re}\int_\gamma g\omega = 0$ for every closed loop on $\hat{S}$. Calculating this integral for simple loops about three of the punctures shows that c is pure imaginary.

Writing $c = \lambda i$, λ real, we obtain

$$\mathrm{Re}\int_{\gamma_1} \phi_1 = \frac{\sin\theta}{\sin 2\theta} \cdot \lambda\pi, \qquad \mathrm{Re}\int_{\gamma_1} \phi_2 = \frac{-\cos\theta}{\sin 2\theta} \cdot \lambda\pi,$$

where γ_1 is a simple loop about $e^{i\theta}$. Also,

$$\operatorname{Re}\int_{\gamma_2}\Phi_1 = \frac{\pi R}{2\sin 2\theta}, \qquad \operatorname{Re}\int_{\gamma_2}\Phi_2 = \frac{\cos 2\theta}{2\sin 2\theta},$$

where γ_2 is a simple loop about $e^{-i\theta}$. This implies that the four sides of P are of length $|\pi\lambda/2\sin 2\theta|$, which completes the proof of 2.1.

III. Intersecting line boundaries. In this section we will consider CMS's with boundary polygonal arcs composed of long and short lines.

Let L_1, L_2 be infinite rays issuing from the origin in the x_1, x_2 plane P. Let A be one of the sectors in P bounded by $L_1 \cup L_2$.

THEOREM III.1. *Let M be a minimal surface with boundary $L_1 \cup L_2$ and* int M *a graph over A. Then $M = A$ (M is a plane).*

Proof. Let $\overline{M}$ be the minimal surface obtained by successive reflections of M through $L_1 \cup L_2 - L_1 \cap L_2$. Let (g, ω) be a Weierstrass representation of $\overline{M}$.

We orient M so that the normal to M points into the southern hemisphere. Then this is true on $\overline{M}$, since the reflections of M are also graphs over sectors in P (the normal may be horizontal at points of $L_1 \cup L_2$). Hence g: $\overline{M} \to \mathbb{C}$ is bounded, and ω is a holomorphic form on $\overline{M}$ with no zeros.

Define a map F: $\overline{M} \to C$ by $F(z) = \int_0^z \omega$, where the integral is taken over any path on M from 0 to z. It follows from the work of Beeson [1] that ω is analytic in the corner, which allows us to integrate ω on a path starting at 0. It is not hard to see that this is independent of the path chosen. We claim F only takes the value 0 at 0. For suppose $F(w) = 0$, $w \neq 0$. We have $F'(w) \neq 0$; hence we can find a local conformal inverse G of F with $G(0) = w$. Let $R > 0$ be the largest R for which G can be defined on $D_R(0)$. If $R = \infty$, the $g \circ G$ is a bounded holomorphic map on C; hence, g is constant and M is a plane. If $R < \infty$, let p be a point of $\partial D_R(0)$, where G cannot be extended to a neighborhood of p. Let $L = \{tp/0 \leqslant t < 1\}$ and $\gamma = G(L)$. Then γ has finite length, since the metric on M is $ds = |\omega|(1 + |g|^2/2)$, $|g|$ is bounded, and

$$\int_\gamma |\omega| = \int_0^1 |f(G(tp))|\,|G'(tp)|\,|p|\,dt = R.$$

Clearly, if γ converged to a point of $\overline{M}$, say z_1, then $F(z_1) = p$ and $F'(z_1) \neq 0$, so G could be extended to a neighborhood of p. So γ must converge to 0. But then $F(0) = p$ and $p \neq 0$, so this is impossible. Hence $F(\overline{M}) \subset C - 0$.

Let p: $C \to C - 0$ be the exponential map, and let F_0: $M \to C$ be a lifting of F. Choose $z_0 \in M$ with $F_0(z_0) = 0$. Let G be a local inverse of F defined in a neighborhood of 0 with $G(0) = z_0$. Let $R > 0$ be the largest number such that G can be defined on $D_R(0)$ to be an inverse of F_0. Exactly as before, one proves

$R = \infty$. Hence $g \circ G$ is a bounded holomorphic map on C and $\overline{M}$ is a plane. (The idea in this proof has already been used by Osserman [4].)

Next we consider a connected boundary composed of two long lines and one short line. Let $L_1 = \{(0, t, 0)/t \geqslant 0\}$, $L_2 = \{(0, 0, t)/0 \leqslant t \leqslant 1\}$, $L_3 = \{(0, t, 1)/t \leqslant 0\}$, and $A = \{(x_1, x_2, 0)/x_1 \geqslant 0\}$.

THEOREM III.2. *Let M be a minimal surface with boundary $L_1 \cup L_2 \cup L_3$, and suppose* int M *is a graph over A. Then M is part of a helicoid.*

Proof. Consider the Gauss map g of M. Along L_2, g is injective (since int M is a graph over A) and takes its values in the equator. There are no other points of M with a horizontal normal vector. This is obvious for points of int M. Suppose $x \in L_1 \cup L_3$ has a horizontal normal vector. Consider the vertical plane P through x containing L_1. M is entirely on one side of P (by the graph hypothesis) and M is tangent to P at x, so by the boundary maximum principle for minimal surfaces, we have $M = P$, a contradiction.

Now let $\overline{M}$ be the minimal surface obtained by reflecting M through L_2. $\overline{M}$ is an embedded CMS with boundary two long lines $\overline{L}_1$, $\overline{L}_3$ (that contain L_1, L_3, respectively). As in the proof of I.1, we see that $\overline{M}$ has finite total curvature: quotient $\overline{M}$ by the translation taking $\overline{L}_1$ to $\overline{L}_3$. The orientable two-sheeted cover of this surface is conformally a two-punctured sphere (its Gauss map covers a set of positive capacity at most twice) and is of finite total curvature. Now the conclusion of III.2 results from the following theorem of E. Toubiana (to appear).

THEOREM III.3. *Let M be an embedded CMS in $\mathbb{R}^3$ with boundary two parallel long lines. If the total curvature of M is finite, then M is part of a helicoid.*

Remarks. (1) Consider the polygon L obtained from $L_1 \cup L_2 \cup L_3$ by rotating L_3 by an angle α in its horizontal plane and leaving L_1, L_2 fixed. Let A be a sector in the (x_1, x_2) plane bounded by L_1 and the projection of (the rotated) L_3. An interesting problem is to determine whether the helicoid is the only minimal surface with boundary L and interior a graph over A. For $\alpha = \pi$, our techniques imply that M is of finite total curvature. However, this does not appear sufficiently conclusive, for the reflection of M through L_2 may not be embedded, so we cannot apply Toubiana's theorem.

(2) Let P be a parallelogram in the (x_1, x_2) plane with sides A_1, B_1, A_2, B_2, and suppose $|A_1| + |A_2| < |B_1| + |B_2|$. Jenkins and Serrin have proved that there exists a unique minimal surface that is a graph over P with asymptotic values $+\infty$ on A_1, A_2, and arbitrary continuous data on B_1, B_2 [2]. Taking the data zero on B_1, B_2, we obtain a minimal surface with boundary L equal to $B_1 \cup B_2$ together with the four vertical half lines issuing from the four points $\partial B_1 \cup \partial B_2$. Now let M be any CMS with boundary L and int M a graph over P. It seems likely that M is the Jenkins–Serrin solution. As in I.1, we can prove that M is of finite total curvature. Reflecting M through each of the lines in L and quotienting by two horizontal translations, we obtain a two-punctured torus. One should be able to analyze the Weierstrass data of this surface.

(3) Suppose M_0 is bounded by parallel straight lines $\{L_i\}$. Consider $N_0 = M_0 \cup \sigma_k M_0$, σ_k the symmetry of $\mathbb{R}^3$ about L_k. Let N be the manifold obtained from N_0 by identifying L_k with $\sigma_1\sigma_k(L_k)$. Since $\tau_k = \sigma_1\sigma_k$ is an orientation-preserving isometry of N_0, the metric on N_0 passes to a metric on N. Also, the (g, ω) of the Weierstrass representation of M are invariant by τ_k, hence pass to N.

The total curvature of N is twice that of M_0. We know N is of finite conformal type (in particular, N is of finite topological type) if $c(N)$ is finite. It follows that $c(M_0)$ is infinite if the number of lines in ∂M_0 is infinite.

Osserman's theory on values of the Gauss map applies to N. In particular, if $c(N) = \infty$, then g takes every value infinitely often, except perhaps for a set of zero capacity.

Now consider a convex polygon P and a Jenkins–Serrin minimal surface M_0 over P; ∂M_0 consists of the vertical lines over the vertices of P, and the previous discussion applies. However, we know that g is injective on each line L_i and that the only points on M_0 with a vertical tangent plane are on ∂M_0. Hence, $c(M_0) < \infty$, since the capacity of the unit circle is positive.

Clearly, $c(M_0)$ is an integral multiple of 2π. In fact, $c(M_0)$ can be calculated by remarking that the normal vector on L_i turns exactly by the angle at the vertex of P defining L_i as one traverses L_i from bottom to top $(-\infty$ to $+\infty)$.

N is conformally a punctured sphere, and the degree of g on the conformal compactification of N is equal to the degree of g on the compactification of ∂M_0 (since g on M_0 points into one hemisphere and on $\sigma_1 M_0$ points into the other). Hence the degree of g is $\frac{1}{2}(n-2)$ and $c(M_0) = (n-2)\pi$, where n is the number of vertices of P.

We remark that the preceding discussion applies when ∂M_0 also contains planar geodesic curves: M_0 extends across the plane of the curve by reflection through the plane.

Added in proof: R. Krust and P. Collin have removed the hypothesis of finite total curvature in I.2.

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