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# SYMMETRY OF CONSTANT MEAN CURVATURE HYPERSURFACES IN HYPERBOLIC SPACE

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**Introduction.** In a recent paper, M. Do Carmo and B. Lawson studied hypersurfaces $M$ of constant mean curvature in hyperbolic space [2]. They use the Alexandrov reflection technique to study $M$ given the asymptotic boundary $\partial_\infty M$. For example, one of their theorems says $M$ is a horosphere when $\partial_\infty M$ reduces to a point. They also prove a Bernstein type theorem for minimal graphs.

In this paper we shall extend their results to other boundary conditions. We prove an embedded $M$, of constant mean curvature, with $\partial_\infty M$ a subset of a codimension one sphere $S$, either is invariant by reflection in the hyperbolic hyperplane $H$ spanned by $S$ or is a hypersphere. In the former case $M$ is a "bigraph" over $H$: it meets any geodesic orthogonal to $H$ either not at all or transversely in two points (one on each side of $H$) or tangentially on $H$.

As a corollary of this, when $\partial_\infty(M)$ consists of two points $p$ and $q$, then $M$ is a hypersurface of revolution about the geodesic joining $p$ to $q$.

We also consider minimal immersed hypersurfaces $M \subset H^n$ with $M$ regular at $\infty$. When $\partial_\infty M$ consists of two disjoint spheres $S_1, S_2$ we prove $M$ is a catenoid or $M$ is the union of the two hyperbolic planes spanned by $S_1$ and $S_2$.

The principal techniques we use to obtain these results are the Alexandrov reflection principle and R. Schoen's adaptation of this to complete minimal surfaces [4].

**I. Definitions and notations.** When we refer to plane, distance, line, etc. we always mean the hyperbolic object in $H^n$. We work with the *Poincaré model* of $H^n$: $H^n$ is the interior of the unit ball in $R^n$. The asymptotic boundary of $H^n$ is identified with the boundary of the unit ball and denoted by $S(\infty)$. Given $A \subset H^n$, we denote by $\partial_\infty A$ the set of accumulation points of $A$ in $S(\infty)$ and call it the *asymptotic boundary* of $A$. When the context is clear, we will omit the subscript $\infty$.

Fix a hyperplane $P_0$ in $H^n$. We have two natural coordinate systems. First, one can use the geodesics orthogonal to $P_0$ to give each point coordinates $(x,t)$ where $x \in P_0$ and $t$ is the distance from $x$ to $(x,t)$. This system does not suit our purposes because translation along one geodesic orthogonal to $P_0$ does not leave invariant another such geodesic. Also this does not extend to a coordinate system on $S(\infty)$.

Instead we shall use the *latitude-longitude system*. More precisely, choose

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coordinates in $P_0$ and let $\gamma$ be the geodesic orthogonal to $P_0$ at an origin $0 \in P_0$. Let $\gamma_t$ be the 1-parameter group of isometries of $H^n$ which along $\gamma$ is translation by a distance $t$ and such that the curves $t \to \gamma_t(x)$ are orthogonal to $P_0$ for each $x \in P_0$ (a positive sense along $\gamma$ is chosen once and for all). Then each point of $H^n$ has coordinates $(x, t)$ where $x \in P_0$ and $\gamma_t(x) = (x, t)$.

Denote by $P_t$ the plane $\gamma_t(P_0)$. We refer to $P_t$ as a *horizontal plane* and the curve $t \to \gamma_t(x)$ as the *vertical curve* through $x$. Notice that for each $s$ the reflection of $H^n$ through the plane $P_s$ is given by the formula $(x, t) \to (x, 2s - t)$.

Let $S_t = \partial_\infty P_t$. Then the coordinate system $(x, t)$ extends to a coordinate system on $S(\infty)$ where each point (except the two limit points of $\gamma$) has a unique coordinate $(x, t)$, $x \in S_0$, $t \in R$. By a Moebius transformation we can send $\gamma$ to the north pole–south pole geodesic and $P_0$ to the equatorial plane. Then the coordinates on $S(\infty)$ are the usual latitude-longitude coordinates.

We say $A \subset H^n$ is a *graph* over $P_s$ if the vertical projection of $A$ to $P_s$ is injective, and $A$ has *locally bounded slope* if the vertical field $v = (0, 1)$ is not tangent to $A$ at any interior point of $A$.

We say $A$ is *above* $B$, $A \geqslant B$, if whenever a vertical curve meets both $A$ and $B$, then every point of $A$ (on this vertical) is above every point of $B$. These notions extend directly to $S(\infty)$ with respect to the horizontals $S_t$ and the vertical curves.

For $A \subset H^n \cup S(\infty)$ and $s \in R$, let $A_{s^+} = \{(x, t) \in A \mid t \geqslant s\}$ and similarly let $A_{s^-}$ be the set of points of $A$ below $P_s$. Let $A^*_{s^+} = \{(x, 2s - t) \mid (x, t) \in A_{s^+}\}$. Also let $H_{s^+}$ (resp. $H_{s^-}$) be the set of all points above $P_s$ (resp. below $P_s$).

A complete hypersurface $M$ is a *hypersphere* over $P_0$ if all points of $M$ lie at the same distance from $P_0$.

**II. The main result.** Now let $M$ be a complete properly embedded hypersurface in $H^n$ of constant mean curvature. Our main result is:

THEOREM 2.1. *If $\partial_\infty M \subset S_0 = \partial_\infty(P_0)$, then $M$ is a hypersphere (in which case $\partial_\infty M = S_0$) or $M$ is invariant under the reflection through $P_0$ and $M$ is a bigraph over $P_0$* (as defined in the introduction).

*Remarks.*

1. It would be interesting to have examples of $M$ as in 2.1, having prescribed boundary. For example, given 3 (or $n$) points on the equator, does there exist a constant mean curvature $M$ with boundary these points?

2. Suppose $\partial_\infty M = S_0$ and $M$ satisfies 2.1. Then is $M$ a hypersphere? Do Carmo and Lawson claimed this in a preliminary version of [2], however, their proof assumed $S_0$ is the homological boundary of $M$. Since our paper was first written, J. Gomes has shown the result to be false without this additional assumption.

We shall use the following version of the *maximum principle* [4]:

1. Let $M_1, M_2$ be connected complete hypersurfaces of constant mean curvatures $C_1, C_2$. Suppose $M_1$ and $M_2$ are tangent at an interior point $x$ and their mean curvature vectors both point in the vertical up direction. If $M_1$ is above $M_2$ in a neighborhood of $x$ and $C_2 \geqslant C_1$, then $M_1 = M_2$.

2. Assume $x$ is an interior point of $\partial M_1$ and $\partial M_2$, $\partial M_1, \partial M_2$ are tangent at $x$ and $M_1, M_2$ as well. Also suppose the mean curvature vectors of $M_1, M_2$ point in the vertical up direction at $x$. If $M_1$ is above $M_2$ in a neighborhood of $x$ and $C_2 \geqslant C_1$, then $M_1 = M_2$ near $x$ (and $M_1, M_2$ are analytic continuations of each other).

*Proof of* 2.1. Let $C$ and $C'$ be the connected components of $H^n - M$ and assume the mean curvature vector $X$ of $M$ points into $C$. Clearly we can suppose $X \neq 0$ since $X = 0$ easily implies $M = P_0$. So there are points of $M$ strictly above or below $P_0$; we can assume above. For $t$ sufficiently large, $P_t$ is disjoint from $M$, so there is a largest $T > 0$ such that $P_T \cap M \neq \emptyset$.

Let $J = \{t \in [0, T] \mid M_{t^+}$ is a graph of locally bounded slope over $P_t$, $M_{t^+}^* \geqslant M_{t^-}$, and for $s \geqslant t$, $X$ points into $H_{s^-}$ at each point of $M \cap P_s\}$. We have $T \in J$ and if $t \in J$, $t \leqslant s \leqslant T$, then $s \in J$. We shall prove $0 \in J$ by showing $J$ is open and closed in $[0, T]$.

First we see why $J$ is *closed*. Suppose $(t, T] \subset J$. If $M_{t^+}$ is not a graph then two points of $M_{t^+}$ are on the same vertical, so there is an $s$, $t < s \leqslant T$, and $x \in P_0$, such that $(x, t)$ and $(x, s)$ are both in $M$. We choose $s$ so there are no other points of $M$ on the vertical $L$ joining $(x, t)$ to $(x, s)$. Now $M$ is a graph in a neighborhood of $(x, s)$, never vertical, and $X$ points into $H_{s^-}$ at $(x, s)$. This implies $L \subset C$ and $M$ is tangent to $P_t$ at $(x, t)$ and below $P_t$ in a neighborhood of $(x, t)$. The reason for the latter property is the vertical curves meeting $M$ in a neighborhood of $(x, s)$, descend to fill a neighborhood of $(x, t)$ in $P_t$. So if any point of $M$ near $(x, t)$ were strictly above $P_t$ there would be some $\tau$ strictly larger than $t$ for which $M_{\tau^+}$ is not a graph. This violates $(t, T] \subset J$.

Now since $M$ is entirely below $P_t$ in a neighborhood of $(x, t)$, $X$ must point into $H_{t^-}$ at $(x, t)$. But $L \subset C$ and $X$ points into $C$ along $M$ so this is a contradiction, and $M_{t^+}$ is a graph over $P_t$. $M_{t^+}$ has locally bounded slope as well. Also $M_{t^+}^* \geqslant M_{t^-}$ since were this not the case, it would already fail to be true for $s$ slightly larger than $t$.

Finally, the mean curvature vector $X$ points into $H_{t^-}$ at each point of $M \cap P_t$, since at $(x, t) \in M$, $M$ is either entirely below $P_t$ at $(x, t)$ (in which case $X$ points down) or $(x, t)$ is an accumulation point of points of $M$ above $P_t$, and then $X$ points into $H_{t^-}$ by continuity. Notice that our convention allows $X$ to be tangent to $P_t$ and point into $H_{t^-}$. So we have proved $J$ is closed.

Next we show $J$ is *open*. Let $[t, T] \subset J$ with $t > 0$. Let $(x, t) \in M$ and let $D \subset M$ be a disc containing $(x, t)$. Notice that $M$ is not vertical at $(x, t)$, for if this were so, consider the half discs $D_{t^+}^*$ and $D_{t^-}$. They have the same boundary, they are tangent at $(x, t)$ and their mean curvature vectors are the same at $(x, t)$. Moreover, $D_{t^+}^*$ is not vertical at points strictly below $P_t$ and $D_{t^+}^* \geqslant D_{t^-}$, hence they do not cross at $(x,t)$. So by the maximum principle, $D^*_{t^+} = D_{t^-}$ near $(x,t)$ and by analytic continuation $M^*_{t^+} = M_{t^-}$. But $t > 0$, so this contradicts $\partial M \subset S_0$.

This proves $M$ is a graph in a neighborhood $U$ of $P_t$ and not vertical in $U$. This implies $X$ points down in $U$ as well (for $s \in (0,t)$, the part of $M$ between $P_s$ and $P_t$ is compact). It remains to verify that $M^*_{s^+} \geqslant M_{s^-}$ for $s$ near $t$. This is done exactly as in [4] so we just sketch the argument here. Since $M \cap U$ is a graph, we have $M^*_{s^+} - V \geqslant M_{s^-}$ for $V$ a neighborhood of $P_t$, $V \subset U$, and $s$ near $t$. Also $M_{s^+} - V$ is compact and its image under reflection through $P_t$ is disjoint from $M_{t^-}$, so by continuity, for $s$ near $t$, we have $(M^*_{s^+} - V) \geqslant M_{s^-}$. This means $M^*_{s^+} \geqslant M_{s^-}$ for $s$ near $t$.

This argument uses the fact that $M_{t^+}$ is strictly above $M_{t^-}$ whenever $(t, T] \subset J$ and $t \neq 0$. If not, there would be a largest $t_0 \geqslant t$ for which this fails. Then $M^*_{t_0^+}$ and $M_{t_0^-}$ are tangent at some point $q$ and their mean curvature vectors have the same direction at $q$ ($M$ separates into two connected components and if $X$ points into $C$ then the mean curvature vector of $M_{t_0^+}$ also points into $C$). Thus $M$ would be invariant under reflection through $P_{t_0}$ by the maximum principle and this violates the hypothesis on $\partial_\infty M$. Thus we have proved $0 \in J$; in particular, $X$ points into $H_{0^-}$ at each point of $M \cap P_0$.

Suppose there are points of $M$ strictly below $P_0$. Then the same reasoning used to prove $J = [0, T]$ shows $M_{0^-}$ is a graph of bounded slope and $X$ points into $H_{0^+}$ at each point of $M \cap P_0$ (just turn $H^n$ over). Hence $X$ is horizontal along $M \cap P_0$, or equivalently $M$ is vertical at each $x \in M \cap P_0$. Let $x \in M \cap P_0$ and $D$ be a disc in $M$ containing $x$. Since $D^*_{0^+} \geqslant D_{0^-}$ and $D$ is vertical at $x$, not vertical at each $y \in D \cap P_s$, $s > 0$, we conclude, as before, $D^* = D$ and hence $M^* = M$ by analytic continuation. Thus when $M$ has points on both sides of $P_0$ we know $M$ is *invariant by reflection* through $P_0$. Since $\gamma$ was an arbitrary geodesic orthogonal to $P_0$, we also know $M$ is a *bigraph* over $P_0$.

To complete the proof of theorem 2.1, we must show that if $M \subset H_{0^+}$ then $M$ is a *hypersphere*. Following [2], we first show the mean curvature $H$ of $M$ must be between 0 and 1: $0 < H < 1$. Consider the family $L_\tau$ of horospheres tangent to $S(\infty)$ at the south pole, the parameter $\tau$ chosen so that $L_0 =$ the south pole and $L_\infty = S(\infty)$. For small positive $\tau$, $L_\tau$ is disjoint from $M$ so there is a smallest $\tau$ such that $L_\tau \cap M \neq \emptyset$. Let $x$ be such an intersection point; clearly $x \in P_s \cap M$, $s > 0$, so $X$ points into $H_{s^-}$ at $x$. Now the mean curvature vector $Y$ of $L_\tau$ at $x$ also points into $H_{s^-}$ and $M$ and $L_\tau$ are tangent at $x$ so $X$ and $Y$ have the same direction at $x$. Therefore $H$ is less than the mean curvature of $L_\tau$, which is one, and we have strict inequality by the maximum principle.

Let $N_0$ be the hypersphere of $H^n$ with boundary $\partial P_0$ and with mean curvature vector pointing down and of length $H$. Consider the family of hyperspheres $N_t = \gamma_t N_0$. We claim $M = N_0$. If not, there are points of $M$ above or below $N_0$. Suppose there are points below. Then there is a smallest negative $t$ such that $N_t \cap M \neq \emptyset$. $N_t$ is tangent to $M$, on one side of $M$, and their mean curvature vectors have the same orientation. So by the maximum principle $M = N_t$, which contradicts $\partial M \subset S_0$. A similar argument works if there are points of $M$ above $N_0$. Q.E.D.

COROLLARY 2.2. *In addition to the hypothesis of* 2.1, *assume* $\partial_\infty M$ *consists of two distinct points* $p$ *and* $q$. *Then* $M$ *is a hypersurface of revolution about the geodesic* $\Gamma$ *joining* $p$ *to* $q$. *More precisely, let* $Q$ *be any hyperplane orthogonal to* $\Gamma$. *Then* $M$ *is transverse to* $Q$ *and* $M \cap Q$ *is a round sphere centered at* $\Gamma \cap Q$.

*Remark*. These hypersurfaces of revolution have been classified by Hsiang [3], and in this paper he obtains a special case of corollary 2.2.

*Proof of* 2.2. It follows immediately from 2.1 that $M$ is transverse to $Q$ except possibly at $y = \Gamma \cap Q$, and that $M \cap Q$ is either the empty set, or $\{y\}$, or a round sphere centered at $y$. It cannot be empty since $Q$ separates $p$ and $q$. If $M \cap Q = \{y\}$, then $M$ is tangent to $Q$ at $y$ and lies on one side of $Q$ near $y$. It follows that hyperplanes $Q'$ close to $Q$, on the other side, do not meet $M$. This is a contradiction, so $M \cap Q$ is a round sphere.

**III. Minimal hypersurfaces of** $H^n$. Let $M$ be a complete minimal hypersurface of $H^n$. We say that $M$ is *regular at* $\infty$ if the asymptotic boundary $B$ of $M$ is a $C^2$ codimension one submanifold of $S(\infty)$ and $\overline{M} = M \cup B$ is of class $C^1$ on $B$. M. Anderson has proved that any $C^2$ codimension one submanifold $B \subset S(\infty)$ bounds a minimal $M \subset H^n$ [1]. We do not know if one has boundary regularity as in the euclidean category (this seems likely for area-minimizing minimal hypersurfaces). In this section we adapt the work of R. Schoen to our context to obtain information about $M$ given $B$.

THEOREM 3.1. *Let* $B \subset S(\infty)$ *be a* $C^2$ *codimension one immersed boundary, not necessarily connected. Assume* $B_{0^+}$ *is a graph of locally bounded slope and* $B^*_{0^+} \geqslant B_{0^-}$. *Let* $M$ *be a minimal hypersurface immersed in* $H^n$ *with* $\partial M = B$ *and* $M$ *regular at* $\infty$. *Then* $M_{0^+}$ *is a graph of locally bounded slope and* $M^*_{0^+} \geqslant M_{0^-}$.

*Proof*. First we remark that $M$ is *orthogonal* to $S(\infty)$ along $B$. We see this as follows (assuming for simplicity that $B$ is embedded). Let $x \in B$, and $S_1, S_2 \subset S(\infty)$ be round codimension one spheres passing through $x$ such that each $S_i$ is tangent to $B$ at $x$ and $S_i$ and $B$ are on one side of each other at $x$, and $S_1$ and $S_2$ are on opposite sides of $B$ at $x$ (this is where we need $B$ to be $C^2$). Let $D_1, D_2$ be the disks on $S(\infty)$ with boundary $S_1, S_2$. Choose the $S_k$ small enough so that $\operatorname{int} D_k$ is disjoint from $B$ for $k = 1,2$. Let $y \in \operatorname{int} D_k$, and $S(y)$ be a small round sphere centered at $y$. Let $H(y)$ be the hyperbolic plane of $H^n$ with boundary $S(y)$. For $S(y)$ small, $H(y)$ is disjoint from $M$. As $S(y)$ grows to become $S_k$, $H(y)$ stays disjoint from $M$ since if this were not so we would have $M = H(y)$ by the maximum principle. Thus $M$ is forced between the two hyperbolic planes $H_1, H_2$ with boundaries $S_1, S_2$, and is orthogonal to $B$.

Let $T > 0$ be the largest $T$ such that $P_T \cap B \neq \emptyset$. Let $J = \{t \in [0, T] \mid M_{t^+}$ is a graph of locally bounded slope and $M^*_{t^+} \geqslant M_{t^-}\}$. As in the proof of 2.1, we see that $J$ is open and closed in $[0, T]$, hence $0 \in J$ and 3.1 is proved. We will not go through the details, however some comments are in order. $M$ need not be embedded to make the argument since the maximum principle applies without worrying about local orientations. Also $M$ orthogonal to $S(\infty)$ along $B$ implies $M$ is a graph of locally bounded slope in a neighborhood of $B_{0^+}$. One can prove, as in theorem 2 of [4], that if $B$ is embedded and $B_{0^+}^* = B_{0^-}$ then $M$ is embedded and $M^* = M$.

THEOREM 3.2. *Let $S_1, S_2$ be disjoint round spheres in $S(\infty)$ and let $M$ be a connected minimal hypersurface immersed in $H^n$ with $\partial_\infty M = S_1 \cup S_2$ and $M$ regular at $\infty$. Then $M$ is a catenoid (i.e., $M$ is embedded and $M$ is a hypersurface of revolution about a geodesic).*

*Remark.* By a conformal transformation of $S(\infty)$ we can make $S_1$ and $S_2$ horizontal and symmetric with respect to the equatorial plane (send the geodesic joining the centers of $S_1$ and $S_2$ to the north pole–south pole geodesic). If $S_1$ and $S_2$ are close enough then there exist exactly two catenoids with boundary $S_1 \cup S_2$ (this can be deduced from [3]). If $S_1$ and $S_2$ are too far apart, there is no catenoid joining them; thus the only minimal hypersurface bounded by $S_1 \cup S_2$ is the union of the two hyperplanes spanned by $S_1$ and $S_2$.

*Proof of* 3.2. We can assume $S_1, S_2$ are horizontal and symmetric with respect to the equator. Then by 3.1, $M$ is also symmetric with respect to the equatorial plane. It suffices to show $M$ is a surface of revolution about the geodesic $\gamma$ joining the north and south poles. Let $P$ be a hyperplane containing $\gamma$; we need show $M$ is invariant by reflection in $P$. Rotate $H^n$ by $\pi/2$ so that $P$ becomes horizontal. Then $B = S_1 \cup S_2$ satisfies the hypothesis of 3.1 from above and below $P$ so $M$ is invariant by reflection through $P$.

THEOREM 3.3. *Let $B \subset S(\infty)$ be a $C^2$ codimension one submanifold and suppose $B$ is a graph of locally bounded slope over $S_0$. Let $M$ be a hypersurface of constant mean curvature $H$ embedded in $H^n$ with the homological boundary of $M$ equal to $B$. Then $M$ is a graph over $P_0$, and if $H \neq 0$ then $M$ is uniquely determined by the value of $H$ and the component of $H^n - M$ into which $X$ points (top or bottom). If $M$ is minimal, then $M$ is a graph and is unique, even if $M$ is only assumed to be immersed with $\partial_\infty M = B$.*

*Remark.* The existence of a minimal $M$ with $\partial_\infty M = B$ is proved in [1]. The unicity of a minimal embedded $M$ with $\partial_\infty M = B$ is in [2].

*Proof of* 3.3. It suffices to show that, if $M$ and $M'$ are as in 3.3 and have the same curvature $H$ (with $X$ and $X'$ pointing the same way if $H \neq 0$), then $\gamma_t M \cap M' = \emptyset$ for $t \neq 0$.

Suppose on the contrary that $\gamma_t M \cap M' \neq \emptyset$ for some nonzero $t$ (say $t > 0$). Let $T$ be the largest $t$ with this property. Then at any $p \in \gamma_T M \cap M'$ the surfaces $\gamma_T M$ and $M'$ are tangent, are on one side of each other, and have the same mean curvature vector (because on the vertical curve through $p$ there is no point of $M'$

above $p$ and no point of $\gamma_T M$ below $p$). By the maximum principle $\gamma_T M = M'$, hence $T = 0$.

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