Topology Vol. 22, No. 2, pp. 119–135, 1983
Printed in Great Britain.

0040-9383/83/020119-17$03.00/0
Pergamon Press Ltd.

FOLIATIONS AND LAMINATIONS ON HYPERBOLIC SURFACES

GILBERT LEVITT

(*Received* 18 *February* 1981)

§1. INTRODUCTION AND STATEMENT OF RESULTS

GEODESIC LAMINATIONS on surfaces have been introduced by Thurston in his study[17] of 3-manifolds with a hyperbolic structure (i.e. a Riemannian metric of constant curvature -1). He has also noticed (unpublished) that, given a hyperbolic structure on a surface, there is a bijective correspondence between measured laminations and equivalence classes of measured foliations (a related result has been obtained by Aranson and Grines[1]).

But a general foliation with p-prong saddle singularities ($p \geq 3$) on a surface is topologically conjugate to a measured foliation (in the sense of [4, 16]) if and only if its non-compact leaves are locally dense (see §4 below), and very few foliations (even of class C^∞) satisfy this condition. The aim of the present paper is to show how Thurston's construction can be generalized to non-measured foliations in order to yield a canonical representation of foliations with saddle singularities on a given hyperbolic surface.

Let us first consider (non-singular) foliations $\mathcal{F}$ with no compact leaf on the 2-torus $T^2 = R^2/Z^2$. As is well known since Denjoy[3], such a foliation, if C^2, is topologically conjugate to an "irrational flow" (the projection onto T^2 of a foliation of R^2 by lines of irrational slope). Denjoy also gave an example of a C^0 (even C^1) foliation not conjugate to an irrational flow. Roughly speaking, his example can be obtained from an irrational flow by "opening up" a leaf f, that is replacing f by two leaves f^+ and f^- whose distance goes to 0 as one goes out to infinity on either leaf, and pushing apart the other leaves to make room; the space between f^+ and f^- is then filled in by new leaves, and the complement of these new leaves is a non-trivial compact minimal set.

One way of distinguishing a Denjoy example from a C^2 foliation is by considering the induced foliation $\tilde{\mathcal{F}}$ on the universal covering R^2. Given a leaf $\tilde{f}$ of $\tilde{\mathcal{F}}$, the line defined by two points p and q of $\tilde{f}$ has a limiting position $\gamma(\tilde{f})$ when p and q go to infinity in opposite directions on $\tilde{f}$. The line $\gamma(\tilde{f})$ has some irrational slope α, and conversely every line of slope α is obtained by "straightening" a leaf $\tilde{f}$. If $\mathcal{F}$ is conjugate to an irrational flow, different leaves of $\tilde{\mathcal{F}}$ give rise to different lines. If $\mathcal{F}$ is a Denjoy example, corresponding lifts to R^2 of the leaves f^+ and f^- (and all "new" leaves in between) give rise to the same line. We call such a line "thick".

Projecting down to T^2, we see that by straightening leaves we have attached to a foliation $\mathcal{F}$ an irrational flow $\gamma(\mathcal{F})$, which is a totally geodesic foliation for the canonical flat metric of T^2; if $\mathcal{F}$ is not conjugate to an irrational flow, then $\gamma(\mathcal{F})$ has at least one thick leaf. Markley has shown[8] that foliations on T^2 with no compact leaf can be classified up to isotopy by specifying an irrational flow on T^2 and a family (at most countably infinite) of thick lines; the foliation is obtained from the irrational flow by opening up these thick lines.

A similar pattern works for a foliation $\mathcal{F}$ with saddle singularities on a hyperbolic surface M, except that by straightening leaves of $\mathcal{F}$ we get a *geodesic lamination* instead of a geodesic foliation; a geodesic lamination is a non-empty set of disjoint simple geodesics (the leaves of the lamination) whose union is closed in M (see §2 below).

Let us be more precise. We consider a closed orientable surface M of genus ν greater than 1 equipped with a fixed hyperbolic metric, and foliations $\mathcal{F}$ on M whose singularities are saddles with any number $p \geq 3$ of prongs. We shall always assume that $\mathcal{F}$ satisfies the following technical condition (*): "if f and f' are homotopic (possibly singular) compact leaves of $\mathcal{F}$, all leaves contained in the open annulus bounded by f and f' are also compact" (see the precise definition of a singular compact leaf and its homotopy class in §3 below); in particular $\mathcal{F}$ has no Reeb component. This is no significant restriction, as any foliation can be reduced to a foliation of this type by modifying it in finitely many annuli bounded by compact leaves; the restriction of the original foliation to these annuli is very easy to describe (see, e.g. [5], §1).

Two foliations will be considered *equivalent* if one can pass from one to the other by Whitehead operations (see [4, 16]) and an *isotopy* (= conjugation by a homeomorphism isotopic to the identity).

We can now state the following result:

THEOREM 1. *Let M be a closed orientable hyperbolic surface.*

(1.1) *To a foliation $\mathcal{F}$ on M are canonically associated a geodesic lamination $\gamma(\mathcal{F})$ and a family $e(\mathcal{F})$ of leaves of this lamination; this family is at most countably infinite and contains all isolated leaves of $\gamma(\mathcal{F})$. If $\mathcal{F}$ and $\mathcal{F}'$ are equivalent foliations, then $\gamma(\mathcal{F}) = \gamma(\mathcal{F}')$ and $e(\mathcal{F}) = e(\mathcal{F}')$.*

(1.2) *Given a lamination γ and a family e of leaves of γ as above, there exists a foliation $\mathcal{F}$ such that $\gamma(\mathcal{F}) = \gamma$ and $e(\mathcal{F}) = e$. This foliation is unique up to equivalence.*

Basic properties of a foliation $\mathcal{F}$ can easily be expressed in terms of $\gamma(\mathcal{F})$ and $e(\mathcal{F})$. For example:

THEOREM 2. *Let M be as in Theorem* 1.

(2.1) *If $\mathcal{F}$ is topologically conjugate to a measured foliation* (i.e. if there exists a transverse measure whose support is all of M), *then $e(\mathcal{F})$ consists exactly of the compact leaves of $\gamma(\mathcal{F})$, and conversely* (examples with $e(\mathcal{F})$ containing non-compact geodesics will be given in §5 below).

(2.2) *Given a transverse measure μ of $\mathcal{F}$, there exists a unique transverse measure μ' of $\gamma(\mathcal{F})$ satisfying $i(\mathcal{F}, \mu; C) = i(\gamma(\mathcal{F}), \mu'; C)$ for all isotopy classes $C \in \mathcal{S}$ of simple closed curves* (the number $i(\mathcal{F}, \mu; C)$ is defined as in [4 or 16]; define $i(\gamma(\mathcal{F}), \mu'; C)$ as the mass deposited by μ' on the geodesic representing C).

(2.3) *A foliation $\mathcal{F}$ contains a leaf cycle if and only is $M - \gamma(\mathcal{F})$ is not simply connected.*

(2.4) *The lamination $\gamma(\mathcal{F})$ contains a compact geodesic g if and only if $\mathcal{F}$ contains a (possibly singular) compact leaf homotopic to g.*

Once a hyperbolic structure has been chosen on M, this correspondence between foliations and laminations yields a canonical way of representing an equivalence class of foliations. This can be used for instance to study transversality of foliations, or to define the intersection number of two measured foliations (Thurston). It is also possible, using laminations, to construct many examples of foliations with no compact leaf, no leaf cycle, and no dense leaf.

When studying transverse measures of foliations, assertions 1.1, 1.2, 2.1 and 2.2 show that there is no loss of generality in restricting oneself to "measured foliations". The topology and dynamics of a foliation $\mathscr{F}$, however, depend heavily on the leaves in $e(\mathscr{F})$; in [6, 7], we shall study the dynamics of $\mathscr{F}$, in particular its limit sets, and explain how they can be interpreted in terms of $\gamma(\mathscr{F})$ and its sublaminations.

I am very grateful to François Laudenbach and Steve Kerckhoff for helping me understand Thurston's construction. I also wish to thank the university of California at Berkeley for its hospitality during the preparation of this paper, and the referee for helpful comments.

We shall now proceed to the proofs. Assertion 1.1 will be proved in paragraph 3, assertion 1.2 in paragraphs 6 and 7. Theorem 2 will be proved in paragraph 4. Paragraph 2 is devoted to geodesic laminations, paragraph 5 to examples, and paragraph 8 to the "extension lemma".

§2. GEODESIC LAMINATIONS

A *geodesic lamination* on a closed hyperbolic surface of genus $\nu \geq 2$ is a non-empty collection γ of disjoint simple (= not self-intersecting) geodesics of M whose union is closed in M. This closed set is the *support* of γ; it always has area 0 in M ([17] p. 8.27; see §4 below), in particular it is nowhere dense. The support therefore determines the lamination, and we can confuse a lamination and its support. The geodesics in γ are called *leaves* of γ.

A *sublamination of* γ is a lamination contained in γ. A lamination is *minimal* if it contains no sublamination (except itself). A geodesic g in γ is *proper* (resp. *isolated*) if there exists an open transverse interval meeting g exactly once (resp. meeting g but no other leaf of γ). The geodesic g is *isolated on one side* in γ if there exists a transverse interval $[a, b]$ with $a \in g$ but $[a, b] \cap \gamma = \emptyset$. The component of $M - \gamma$ which contains b is *bounded by* g.

Note that an isolated geodesic is proper (and therefore isolated on both sides): if g were isolated but not proper, its intersection with some compact transverse interval would be perfect but countable. The same argument shows that a minimal lamination either consists of a compact leaf or contains uncountably many leaves.

We shall consider the universal covering $p: \tilde{M} \to M$. The space $\tilde{M}$, with the induced Riemannian metric, is isometric to the hyperbolic plane, and via the Poincaré model we identify it with the interior of a disc bounded by a *circle at infinity* S. It is

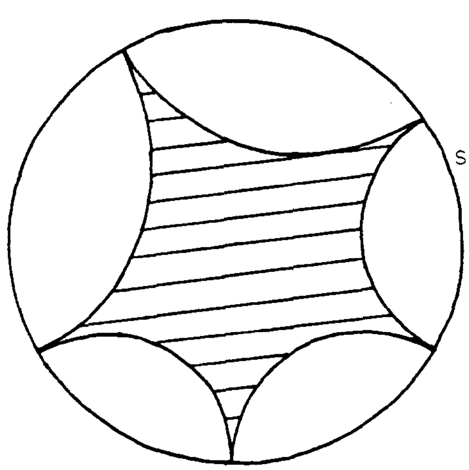


Fig. 1.

well known that any covering transformation $\sigma \in \pi_1 M$ can be extended to a homeomorphism of the closed disc $\tilde{M} \cup S$, and that for the action of $\pi_1 M$ thus induced on S all orbits are dense.

In the Poincaré model, geodesics are contained in circles orthogonal to S (and diameters). Two distinct points of S can be joined by a unique geodesic of $\tilde{M}$. If S' is a finite subset of S consisting of at least 3 points, the closed subset of $\tilde{M}$ bounded by the geodesics joining neighboring points of S' is called an *asymptotic polygon* (see Fig. 1); its area is $(|S'|-2)\pi$.

If γ is a lamination on M, the preimage $\tilde{\gamma} = p^{-1}(\gamma)$ is a lamination on $\tilde{M}$, and it defines a closed $\pi_1 M$-invariant subset $\mathscr{G}_\gamma$ of the space $\mathscr{G}$ of geodesics of $\tilde{M}$ (geodesics of $\tilde{M}$ correspond to pairs of points of S: $\mathscr{G}$ therefore inherits a natural topology).

PROPOSITION. *Let γ be a geodesic lamination on a closed surface M.*

(i) *For a geodesic g in γ, the following conditions are equivalent*: (a) *g is isolated.* (b) *The geodesics in $p^{-1}(g)$ are isolated points of $\mathscr{G}_\gamma$.* (c) *The complement of g in γ is compact* (and therefore is either a lamination or the empty set).

(ii) *If g is not compact, these conditions are also equivalent to the following*: (d) *g is proper.* (e) *g belongs to no minimal sublamination of γ.* (f) *There is no transverse measure of γ whose support contains g.*

(iii) *There are only finitely many isolated leaves*; *the other leaves are partitioned into finitely many minimal sublaminations.*

Remark. Several of the equivalences above are in [17], explicitly or implicitly.

Proof. (i) The equivalence between (a) and (c) is clear. To see that (b)$\Leftrightarrow$(c), note that there is a bijection between sublaminations of γ and closed non-empty $\pi_1 M$-invariant subsets of $\mathscr{G}_\gamma$. If (b) holds, then $\mathscr{G}_\gamma - p^{-1}(g)$ is closed, and $\gamma - g$ is a lamination (unless it is empty). If (c) holds, then $p^{-1}(g)$ is open in $\mathscr{G}_\gamma$. Since g is proper, $p^{-1}(g)$ is discrete and therefore consists of isolated points.

(ii) Here we assume that g is not compact. We have already seen (a)$\Rightarrow$(d), and we shall prove successively (d)$\Rightarrow$(e), (a)$\Rightarrow$(f), (f)$\Rightarrow$(e), (e)$\Rightarrow$(a). In a minimal lamination different from a compact leaf, every half-leaf is dense, and therefore no leaf is proper; this proves (d)$\Rightarrow$(e). To prove (a)$\Rightarrow$(f), simply note that, if a non-compact isolated geodesic g belongs to the support of a transverse measure, then any transverse segment whose interior contains an accumulation point of g has infinite measure, an impossibility. It is easy to check that any lamination admits a transverse measure (obtained as a limit of counting measures, see [17] Proposition 8.10.6). If the lamination is minimal, every leaf is contained in the support. This proves (f)$\Rightarrow$(e).

To prove (e)$\Rightarrow$(a), first consider a component U of $M - \gamma$. The restriction to U of the Riemannian metric of M defines a distance on U, for which we can construct the metric completion $\hat{U}$ ($\hat{U}$ is an abstract space, not a subset of M; if for instance U is simply connected, then $\hat{U}$ is congruent to an asymptotic polygon).

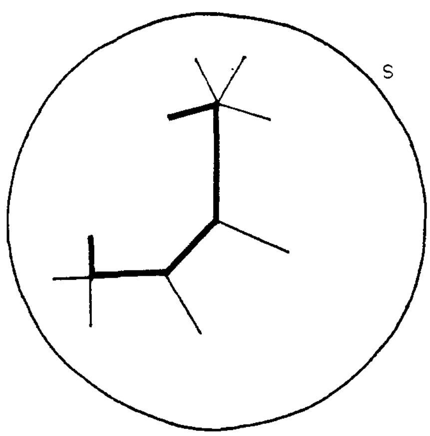
The inclusion map from U to M extends isometrically to an immersion of $\hat{U}$ into M whose image is the union of U and the geodesics of γ that bound it. An easy computation based on the Gauss–Bonnet theorem shows that the area of $\hat{U}$ is an integral multiple $k\pi$ of π, and that the number of geodesics in $\delta\hat{U}$ is at most $k+2$. Noting that $k+2 \leq 3k$ and that M has area $(4\nu-4)\pi$, we see that a lamination contains at most $12\nu - 12$ geodesics isolated on one side.

It follows that γ contains finitely many minimal sublaminations: if $\gamma_1, \ldots, \gamma_q$ are distinct minimal sublaminations of γ, they are disjoint and each contains at least one geodesic isolated on one side in $\gamma_1 \cup \ldots \cup \gamma_q$; therefore $q \leq 12\nu - 12$ (in fact one can prove that γ contains at most $3\nu - 3$ minimal sublaminations). Consequently the union of all minimal sublaminations of γ is a lamination γ_0.

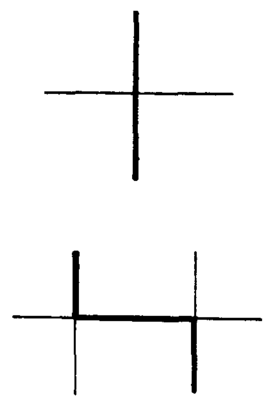
Note that both (ii) and (iii) are now consequences of (e)$\Rightarrow$(a). To prove (e)$\Rightarrow$(a), consider a geodesic $g \in \gamma - \gamma_0$, and denote by U the component of $M - \gamma_0$ that contains g. Since U contains no minimal sublamination of γ, a half-leaf contained in g cannot stay in any compact set $K \subset U$; this can be seen to imply that each end of g must either spiral towards a compact geodesic bounding U or eventually be contained in a cusp of U. Since the total number of cusps of components of $M - \gamma$ is finite, it follows that g is isolated in γ. This completes the proof of the proposition. □

§3. PROOF OF 1.1; CONSTRUCTION OF $\gamma(\mathcal{F})$ AND $e(\mathcal{F})$

Let $\tilde{\mathcal{F}}$ be the foliation induced by $\mathcal{F}$ on $\tilde{M}$. By a *leaf of* $\tilde{\mathcal{F}}$, we shall mean either a regular leaf (containing no singularity), or a connected union h of separatrices satisfying the following condition: if a saddle s belongs to h, exactly two separatrices issued from s belong to h, and these separatrices are adjacent; furthermore separatrices not belonging to h but with an endpoint on h leave h all on the same side (see Fig. 2).



A singular leaf of $\tilde{\mathcal{F}}$



Not a leaf

Fig. 2.

Since all singularities of $\mathcal{F}$ are saddles, a regular compact leaf or a leaf cycle of $\mathcal{F}$ cannot be contractible in M. This implies that a leaf of $\tilde{\mathcal{F}}$ is the image of a continuous injection from R to $\tilde{M}$, and that a separatrix of $\tilde{\mathcal{F}}$ belongs to exactly two leaves.

We define a *leaf of* $\mathcal{F}$ as the projection $f = p(h)$ of a leaf of $\tilde{\mathcal{F}}$. A compact leaf $p(h)$ of $\mathcal{F}$ is therefore either a regular leaf or a union of saddle connections. It cannot be contractible, and so there exists a non-trivial covering transformation sending h into itself. The geodesic of $\tilde{M}$ preserved by this transformation projects to a simple compact geodesic g of M. If $p(h)$ is a regular leaf, it is isotopic to g. If it is singular, we say that it is *homotopic* to g. Note that two homotopic compact leaves always bound an open annulus.

A *transverse curve* is a simple closed curve $C \subset M$ never tangent to $\mathcal{F}$ and containing no singularity of $\mathcal{F}$. Since all singularities of $\mathcal{F}$ are saddles, such a curve is non-contractible (and therefore isotopic to a geodesic). Also note that, if h_0 is a half-leaf of $\tilde{\mathcal{F}}$ such that $p(h_0)$ is not compact and does not spiral towards a compact leaf, then there exists a transverse curve meeting $p(h_0)$ infinitely often: pick a non-compact leaf in $\overline{p(h_0)} - p(h_0)$ and choose any transverse curve meeting it.

The first step in the construction of $\gamma(\mathscr{F})$ is the following result:

LEMMA 1. *Let h be a leaf of $\tilde{\mathscr{F}}$. Each end of h converges to a point of S; the two points at infinity thus determined by h are distinct.*

Proof of Lemma 1. First note that the behavior at infinity of leaves of $\tilde{\mathscr{F}}$ does not change if we replace $\mathscr{F}$ by an isotopic foliation. This is because a homeomorphism φ of M isotopic to the identity can be covered by a homeomorphism $\tilde{\varphi}$ of $\tilde{M}$ such that $\sup_{x \in \tilde{M}} d(x, \tilde{\varphi}x)$ is finite (d denotes distance in $\tilde{M}$); the continuous extension of such a $\tilde{\varphi}$ to $\tilde{M} \cup S$ induces the identity on S. So we can change $\mathscr{F}$ by an isotopy whenever convenient.

Consider an end of h defined by a half-leaf $h_0 \subset h$. The first assertion of lemma 1 is clear if $p(h_0)$ is a compact leaf of $\mathscr{F}$ or spirals towards a compact leaf. Otherwise, there exists a curve C transverse to $\mathscr{F}$ meeting $p(h_0)$ infinitely often; by an isotopy we can assume C is a geodesic.

Since C meets $p(h_0)$ infinitely often, the half-leaf h_0 meets $p^{-1}(C)$ infinitely often. Note however that it can meet a given component at most once: if not, there would exist an angular disc D embedded in $\tilde{M}$, with δD consisting of an arc of h_0 and an arc transverse to $\tilde{\mathscr{F}}$, and this is impossible as all singularities of $\tilde{\mathscr{F}}$ are saddles. All limit points of h_0 in $\tilde{M} \cup S$ therefore belong to the intersection of an infinite family of half-spaces of $\tilde{M} \cup S$ bounded by components of $p^{-1}(C)$.

Because a compact set in $\tilde{M}$ meets only finitely many components of $p^{-1}(C)$, this intersection has to be contained in S. It is connected, and cannot be disconnected by removing an endpoint of a component of $p^{-1}(C)$; since endpoints of components of $p^{-1}(C)$ are dense in S, it follows that h_0 converges to a point of S. We have now proved the first assertion of Lemma 1.

The second assertion is clear if $p(h)$ is compact, or if there exists a transverse curve C meeting $p(h)$ at least twice since then any component of $p^{-1}(C)$ which meets h separates its points at infinity. Otherwise, the ends of $p(h)$ spiral towards compact leaves f_1 and f_2.

Suppose the two points at infinity of h were the same. Then f_1 and f_2 are homotopic (because two non-trivial elements of $\pi_1 M$ having a common fixed point on S preserve the same geodesic). If $f_1 = f_2$, both ends of $p(h)$ spiral towards f_1 on the same side; this is impossible since all singularities of $\mathscr{F}$ are saddles. If $f_1 \neq f_2$, they bound an open annulus containing $p(h)$. This is impossible as $\mathscr{F}$ is assumed to satisfy (*). □

We can now associate to a leaf h of $\tilde{\mathscr{F}}$ the geodesic $\gamma(h)$ joining its two points at infinity. Geodesics $\gamma(h)$ and $\gamma(h')$ attached to leaves h and h' of $\tilde{\mathscr{F}}$ are either equal or disjoint in $\tilde{M}$ (they can have one common endpoint on S). Let $\gamma(\tilde{\mathscr{F}})$ be the union of all the geodesics obtained by this "straightening" process from leaves of $\tilde{\mathscr{F}}$. Note that it is a $\pi_1 M$-invariant subset of $\tilde{M}$. Geodesics in $\gamma(\tilde{\mathscr{F}})$ therefore project onto simple geodesics in M.

If $f = p(h)$ is a leaf of $\mathscr{F}$, we shall denote by $\gamma(f)$ the geodesic $p(\gamma(h))$; it depends only on f, not on the choice of h in $p^{-1}(f)$. Note that if f is compact then $\gamma(f)$ is compact and homotopic to f.

LEMMA 2. *The set $\gamma(\tilde{\mathscr{F}})$ is closed in $\tilde{M}$.*

Before we prove lemma 2, let us use it to define $\gamma(\mathscr{F})$ as the projection $p(\gamma(\tilde{\mathscr{F}}))$; since a set $A \subset M$ is closed if and only if $p^{-1}(A)$ is closed in $\tilde{M}$, this projection is indeed a lamination on M. It is easy to check that $\gamma(\tilde{\mathscr{F}})$ and $\gamma(\mathscr{F})$ depend only on the equivalence class of $\mathscr{F}$.

Proof of Lemma 2. Let $g_n = \gamma(h_n)$ be a sequence of geodesics in $\gamma(\tilde{\mathscr{F}})$ converging to a geodesic g. The g_n's do not intersect g (unless they are equal to g), and we can assume without loss of generality that they are distinct from g and all on the same side of g. Let L be the limit set in $\tilde{M} \cup S$ of the sequence of subsets h_n. For any leaf m of $\tilde{\mathscr{F}}$, the set L meets at most one component of $(\tilde{M} \cup S) - \bar{m}$ (we denote by $\bar{m}$ the closure of m in $\tilde{M} \cup S$, obtained by adding its two points at infinity). Since points at infinity of leaves of $\tilde{\mathscr{F}}$ are dense in S, the set L meets $\tilde{M}$ and therefore contains at least one leaf h of $\tilde{\mathscr{F}}$. Let $h_0 \subset h$ be a half-leaf. In order to prove lemma 2, it suffices to show that h_0 converges to one of the points at infinity of g.

Assume first that there exists a simple closed curve C transverse to $\mathscr{F}$ meeting $p(h_0)$ infinitely often. If h_0 does not converge to the corresponding point at infinity of g, there is a half-space of $\tilde{M} \cup S$ bounded by a component of $p^{-1}(C)$, that contains the point at infinity of h_0 but does not contain the points at infinity of the leaves h_n. This is impossible for n large.

If $p(h_0)$ spirals towards a compact leaf, so do leaves close to it, and h_0 converges to one of the points at infinity of g (which is also a point at infinity of h_n for n large). Finally, if $p(h)$ is compact, then $p(h_n)$ spirals towards it for n large enough; therefore $\gamma(h)$ and g have one point in common at infinity. If $\gamma(h)$ were different from g, applying to h a suitable covering transformation which sends $\gamma(h)$ into itself would yield leaves of $\tilde{\mathscr{F}}$ separating h from the leaves h_n, a contradiction. □

Remark. We have actually proved the following two assertions:

(3.1) *If a sequence h_n of leaves of $\tilde{\mathscr{F}}$ converges towards a leaf h, then $\gamma(h_n)$ converges to $\gamma(h)$.* In particular, the union of all leaves f of $\mathscr{F}$ such that $\gamma(f)$ belongs to a given sublamination of $\gamma(\mathscr{F})$ is compact.

(3.2) *If a sequence $g_n = \gamma(h_n)$ of geodesics in $\gamma(\tilde{\mathscr{F}})$ converges towards a geodesic g, the sequence h_n contains a subsequence converging to a leaf h such that $\gamma(h) = g$.* In particular, pick a leaf f in $\mathscr{F}$; the union of the geodesics $\gamma(f')$, taken over all leaves f' contained in $\bar{f}$, is a sublamination of $\gamma(\mathscr{F})$.

Remark. Say that a compact set $K \subset M$ is *quasiminimal* if it is the closure of a recurrent regular leaf. One can prove that there is a bijective correspondence between quasiminimal sets of $\mathscr{F}$ and minimal sublaminations of $\gamma(\mathscr{F})$ not consisting of a compact leaf (see [6, 7]).

Now we turn to the construction of $e(\mathscr{F})$, a family of leaves of $\gamma(\mathscr{F})$ which is at most countable and contains all isolated leaves of $\gamma(\mathscr{F})$. We say that a geodesic g in $\gamma(\tilde{\mathscr{F}})$ is *thick* if there are two distinct leaves h and h' such that $\gamma(h) = \gamma(h') = g$.

The second assertion of lemma 1 implies that such leaves h and h' are disjoint. They bound in $\tilde{M}$ a connected open set $U(h, h')$, and every leaf m contained in $U(h, h')$ satisfies $\gamma(m) = g$. Therefore any two distinct leaves in $U(h, h')$ are disjoint; in other words, the set $U(h, h')$ contains no singularity of $\tilde{\mathscr{F}}$. Furthermore, we know that the union of all leaves n such that $\gamma(n) = g$ is closed in $\tilde{M}$ (by Assertion 3.1). Consequently there exist extreme leaves h^0 and h^1 (with $\gamma(h^0) = \gamma(h^1) = g$), such that any leaf h with $\gamma(h) = g$ belongs to $h^0 \cup h^1 \cup U(h^0, h^1)$.

The set of thick geodesics in $\gamma(\tilde{\mathscr{F}})$ is preserved by the action of $\pi_1 M$, and its projection is a family $e(\mathscr{F})$ of *thick* geodesics of $\gamma(\mathscr{F})$. This family is at most countable; note that it is not necessarily a sublamination of $\gamma(\mathscr{F})$, as its support need not be closed.

Since obviously $e(\mathscr{F}) = e(\mathscr{F}')$ for equivalent foliations, the last thing we need to check is that all isolated geodesics in $\gamma(\mathscr{F})$ are thick. Let $g \in \gamma(\mathscr{F})$ be isolated, f a leaf of $\mathscr{F}$ such that $\gamma(f) = g$, and h a component of $p^{-1}(f)$. If f is compact, non-thickness of g would imply the existence of leaves of $\mathscr{F}$ spiraling towards f at least on one side of f (on both if f is a regular compact leaf), contradicting the fact that g is isolated.

If f is not compact, consider simple closed curves C and C' transverse to $\mathscr{F}$, and components $\tilde{C}$ and $\tilde{C}'$ of $p^{-1}(C)$ and $p^{-1}(C')$ respectively, such that h meets both $\tilde{C}$ and $\tilde{C}'$; we can choose these curves C, C', $\tilde{C}$ and $\tilde{C}'$ so that any leaf of $\tilde{\mathscr{F}}$ meeting both $\tilde{C}$ and $\tilde{C}'$ has the same points at infinity as h (we use (a)$\Rightarrow$(b) in the proposition in §2). Then leaves m of $\tilde{\mathscr{F}}$ meeting $\tilde{C}$ near $h \cap \tilde{C}$ satisfy $\gamma(m) = \gamma(h)$, and therefore g is thick (if the arc of h between $\tilde{C}$ and $\tilde{C}'$ contains a singularity of $\tilde{\mathscr{F}}$, we consider leaves m only on the side of this arc where there are no separatrices). □

§4. PROOF OF THEOREM 2

Let us restate the assertions contained in Theorem 2:

(2.1) *A foliation $\mathscr{F}$ is topologically conjugate to a measured foliation if and only if $e(\mathscr{F})$ consists exactly of the compact leaves of $\gamma(\mathscr{F})$.*

(2.2) *To a transverse measure μ of $\mathscr{F}$ corresponds a unique transverse measure μ' of $\gamma(\mathscr{F})$ with $i(\mathscr{F}, \mu; C) = i(\gamma(\mathscr{F}), \mu'; C)$ for all isotopy classes $C \in \mathscr{S}$ of simple closed curves.*

(2.3) *A foliation $\mathscr{F}$ contains a leaf cycle if and only if $M - \gamma(\mathscr{F})$ is not simply connected.*

(2.4) *The lamination $\gamma(\mathscr{F})$ contains a compact geodesic g if and only if $\mathscr{F}$ contains a compact leaf homotopic to g.*

We shall prove successively assertions 2.4, 2.3 and 2.1. Proof of Assertion 2.2 is left to the reader.

Proof of 2.4. We have already noted that, if f is a compact leaf of $\mathscr{F}$, then $\gamma(f)$ is compact and homotopic to f. Conversely, let $g \in \gamma(\mathscr{F})$ be compact. Choose a component $\tilde{g}$ of $p^{-1}(g)$, and a non-trivial $\sigma \in \pi_1 M$ with $\sigma(\tilde{g}) = \tilde{g}$. If h^0 is an extreme leaf of $\tilde{\mathscr{F}}$ with $\gamma(h^0) = \tilde{g}$, then $\sigma(h^0) = h^0$, and $p(h^0)$ is a compact leaf of $\mathscr{F}$ homotopic to g. □

Proof of 2.3. Suppose $\mathscr{F}$ contains a leaf cycle. This cycle is not contractible, and the geodesic homotopic to it either is disjoint from $\gamma(\mathscr{F})$ or belongs to $\gamma(\mathscr{F})$ but is isolated on one side (if the cycle is a leaf). In both cases at least one component of $M - \gamma(\mathscr{F})$ is not simply connected.

Now suppose that $\mathscr{F}$ has no leaf cycle. Changing $\mathscr{F}$ by Whitehead operations allows us to assume that there is no saddle connection. If s is a p-prong saddle of $\mathscr{F}$, it belongs to exactly p leaves f_i $(1 \leq i \leq p)$. The geodesics $\gamma(f_i)$ bound a simply connected component U_s of $M - \gamma(\mathscr{F})$ whose area is $(p-2)\cdot\pi$. A p-prong saddle can be viewed as a singularity of index $(2-p)/2$, and the sum of the indices of the singularities of $\mathscr{F}$ is equal to $\chi(M) = 2 - 2\nu$ (see [4], exposé 5, formule I.6). Since the area of M is $(4\nu - 4)\cdot\pi$, every component of $M - \gamma(\mathscr{F})$ is one of the U_s and therefore is simply connected. □

Remark. We have proved in a special case that the support of a lamination has area 0. The proof in the general case is based on a similar computation (see [17], p. 8.27).

Proof of 2.1. Suppose $\mathscr{F}$ is topologically conjugate to a measured foliation. Denote by μ a transverse measure with support equal to M. The existence of μ implies that no leaf of $\mathscr{F}$ can spiral towards a compact leaf; consequently every compact geodesic in $\gamma(\mathscr{F})$ is isolated and thick. We have to show that a non-compact geodesic g in $\gamma(\mathscr{F})$ cannot be thick. Choose a component $\tilde{g}$ of $p^{-1}(g)$, and suppose that h and h' are two distinct leaves of $\tilde{\mathscr{F}}$ with $\gamma(h) = \gamma(h') = \tilde{g}$. We obtain a contradiction as follows.

Since no non-trivial covering transformation sends $\tilde{g}$ into itself, the restriction of the projection p to $U(h, h')$ is injective. The leaf $f = p(h)$ of $\mathscr{F}$ is not compact and does not spiral towards a compact leaf. Therefore there exists a transverse curve C meeting f infinitely often. The intersection of C with the projection of $U(h, h')$ consists of infinitely many disjoint open intervals. Since $U(h, h')$ contains no singularity of $\tilde{\mathscr{F}}$, these intervals all have the same μ-measure. The total μ-measure of C being finite, these intervals are μ-negligible, contradicting the assumption that μ has support equal to M.

Suppose conversely that all geodesics in $e(\mathscr{F})$ are compact. Then every isolated leaf of $\gamma(\mathscr{F})$ is compact, and $\gamma(\mathscr{F})$ is the union of its minimal sublaminations. In order to show that $\mathscr{F}$ is topologically conjugate to a measured foliation, we shall first prove that every non-compact leaf f of $\mathscr{F}$ is locally dense (i.e. its closure has non-empty interior).

Denote by β the closure of $\gamma(f)$ and by β' the complement of β in $\gamma(\mathscr{F})$; note that β' is compact. Let F (resp. F') be the union of all leaves f in $\mathscr{F}$ such that $\gamma(f) \in \beta$ (resp. β'). The sets F and F' are compact (see Assertion 3.1 in §3) and they cover M. Since F' is not equal to M, the set F has non-empty interior, and we only need to show that $\bar{f}$ contains any leaf $m \subset F$.

Consider the subset of $\gamma(\mathscr{F})$ obtained by straightening all leaves in $\bar{f}$. It is compact (§3, Assertion 3.2), and consequently contains β. The set $\bar{f}$ therefore contains a leaf n such that $\gamma(n) = \gamma(m)$. Since $\gamma(m)$ is not thick, we have $n = m$, and finally $m \subset \bar{f}$ as required. Note that, if $\mathscr{F}$ has no leaf cycle, every component of $M - \gamma(\mathscr{F})$ is simply connected and therefore $\gamma(\mathscr{F})$ is a minimal lamination; this implies that every leaf of $\mathscr{F}$ is dense.

What we have done so far proves that in a measured foliation every non-compact leaf is locally dense, and even dense if the foliation has no leaf cycle (one can give a more natural proof, using Poincaré's recurrence theorem; see [4] exposé 9). Conversely, it is a general fact that a foliation whose non-compact leaves are locally dense is topologically conjugate to a measured foliation. Here is a sketch of the proof in the special case when $\mathscr{F}$ has no leaf cycle. One first checks that all leaves of $\mathscr{F}$ are dense. By taking a limit of counting measures, one then proves that $\mathscr{F}$ admits at least one transverse measure. Since leaves are dense, the support of this measure is all of M, and $\mathscr{F}$ is topologically conjugate to a measured foliation. □

Remark. Most results about foliations can be expressed in terms of laminations. For example, Schwartz's theorem ([13]) can be rephrased as follows (see [7]): if $\mathscr{F}$ is defined by a C^2 vector field, then every sublamination of $\gamma(\mathscr{F})$ contains a compact leaf or a non-thick leaf isolated on one side in $\gamma(\mathscr{F})$.

§5. EXAMPLES

For a measured foliation $\mathscr{F}_0$, the family of geodesics $e(\mathscr{F}_0)$ consists exactly of the compact geodesics in $\gamma(\mathscr{F}_0)$. The simplest way to get an example with $e(\mathscr{F})$ containing non-compact geodesics is to start with a measured foliation $\mathscr{F}_0$ and to open up a countable (possibly finite) family of non-compact leaves f_i. The resulting foliation $\mathscr{F}$ satisfies $\gamma(\mathscr{F}) = \gamma(\mathscr{F}_0)$, but $e(\mathscr{F})$ is the union of $e(\mathscr{F}_0)$ and the geodesics $\gamma(f_i)$.

To get other examples, consider a Denjoy foliation $\mathscr{F}_d$ obtained by opening up a leaf f on an irrational flow on T^2, as in §1. Pick a point a^+ on f^+ (resp. a^- on f^-), and take a two-sheeted covering of T^2 branched over a^+ and a^-. The foliation $\mathscr{F}_d$ lifts to an orientable foliation $\mathscr{F}$ on a surface M_2 of genus 2, with two 4-prong saddles s^+ and s^- above a^+ and a^-. The inverse image of the non-trivial compact minimal set of $\mathscr{F}_d$ is the union of two quasiminimal sets K_1 and K_2 of $\mathscr{F}$ whose intersection consists of s^+ and s^- (see remark following Assertion 3.2 in §3 for the definition of the word quasiminimal). They each contain two adjacent separatrices of s^+ and s^-.

Straightening leaves in K_1 (resp. K_2) gives rise to disjoint laminations γ_1 and γ_2; the other leaves of $\mathscr{F}$ get straightened to thick geodesics g and g' whose closures contain γ_1 and γ_2 (see Fig. 3). The lamination $\gamma(\mathscr{F})$ is the union $\gamma_1 \cup \gamma_2 \cup g \cup g'$; it does not carry a transverse measure with full support. The family $e(\mathscr{F})$ consists of g and g'.

Let δ be the compact simple geodesic in $M-(\gamma_1 \cup \gamma_2)$ (see Fig. 3 or 4), and g'' be a simple geodesic in $M-(\gamma_1 \cup \gamma_2 \cup g)$ meeting g' and δ each exactly once (as on Fig. 4). There exists an orientable foliation $\mathscr{F}'$ on M_2 such that $\gamma(\mathscr{F}') = \gamma_1 \cup \gamma_2 \cup g \cup g''$ and $e(\mathscr{F}') = g \cup g''$. It has two quasiminimal sets K_1' and K_2' (one on each side of δ, corresponding to γ_1 and γ_2); if f is a leaf of $\mathscr{F}'$ not contained in $K_1' \cup K_2'$, it meets δ exactly once and its closure is $f \cup K_1' \cup K_2'$.

Unlike $\mathscr{F}$, the foliation $\mathscr{F}'$ can be made transverse to δ by an isotopy, and in fact $\mathscr{F}'$ can be constructed by glueing together along δ two "Cherry examples"; by a Cherry example ([2, 14], Chap. 9) we mean an orientable foliation of the punctured torus, transverse to the boundary, with one 4-prong saddle, no compact leaf and no saddle connection; exactly one separatrix of the saddle reaches the boundary; the other 3, together with the leaves not meeting the boundary, form a quasiminimal set. If the glueing of two Cherry examples is done so as to create one saddle connection (as in [5], Example 3), one gets a foliation $\mathscr{F}''$ for which $e(\mathscr{F}'')$ contains only one geodesic.

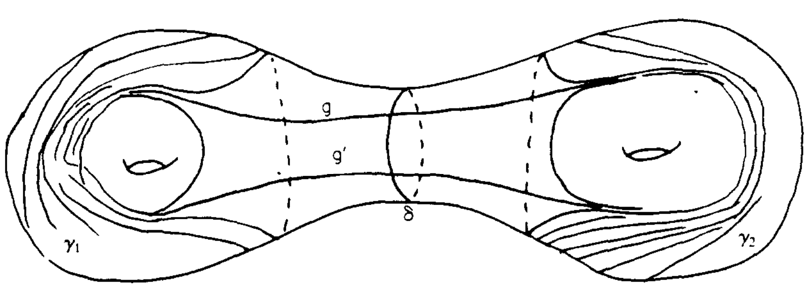
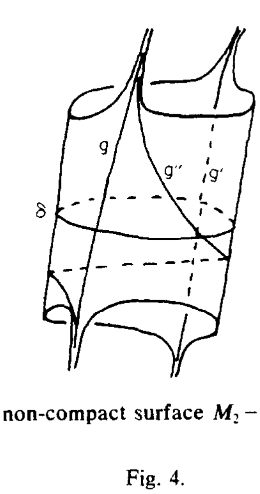


Fig. 3.



The non-compact surface $M_2-(\gamma_1 \cup \gamma_2)$.

Fig. 4.

§6. PROOF OF 1.2: EXISTENCE OF $\mathcal{F}$

Let γ be a geodesic lamination on M, and e a family of leaves of γ, at most countably infinite and containing all isolated leaves of γ. We want to construct a foliation $\mathcal{F}$ with $\gamma(\mathcal{F}) = \gamma$ and $e(\mathcal{F}) = e$.

Here is a rough sketch of the proof. First consider a measured foliation $\mathcal{F}$ with no compact leaf and no saddle connection. We have seen that $e(\mathcal{F})$ is empty and that each component U_s of $M - \gamma(\mathcal{F})$ is congruent to the interior of an asymptotic polygon; its boundary consists of the geodesics obtained by straightening the leaves of $\mathcal{F}$ containing a given saddle s. Conversely, we can pass from $\gamma(\mathcal{F})$ to $\mathcal{F}$ by collapsing each U_s onto the union of s and its separatrices. In other words, there is a continuous surjective map z from M to itself, homotopic to the identity, and carrying leaves of $\gamma(\mathcal{F})$ onto leaves of $\mathcal{F}$. If G is a collection of disjoint compact geodesics g_i separating M into pairs of pants, we can require that z sends each g_i into itself.

Assume for simplicity that γ is a minimal lamination not consisting of a compact geodesic. Using the above observations, we can construct a foliation $\mathcal{F}_0$ with $\gamma(\mathcal{F}_0) = \gamma$ and $e(\mathcal{F}_0) = \emptyset$, as follows: we first choose a pair of pants decomposition G as above; we also choose, for each component U_j of $M - \gamma$, a $1-$complex K_j onto which U_j collapses; we then construct on each g_i a continuous surjective map collapsing to a point each component of $g_i \cap (M - \gamma)$; we finally construct $\mathcal{F}_0$ pair of pants by pair of pants, using these maps and the K_j's. If e is not empty, a foliation $\mathcal{F}$ satisfying $\gamma(\mathcal{F}) = \gamma$ and $e(\mathcal{F}) = e$ can be obtained from $\mathcal{F}_0$ by opening up leaves corresponding to geodesics in e.

We now begin the actual proof. Choose a decomposition of M into pairs of pants by disjoint non-separating compact geodesics g_i $(1 \leq i \leq 3\nu - 3)$ such that no g_i is a leaf of γ and all g_i's meet γ; we shall denote by G the union of the g_i's. The existence of such a decomposition is clear if γ contains a compact geodesic. If not, consider the completion $\hat{U}_j$ of a component U_j of $M - \gamma$ (as in the proof of (e)$\Rightarrow$(a) in §2). A simple computation based on the Gauss-Bonnet theorem shows that the area of $\hat{U}_j$ is at least 2π times its first Betti number. The union of the images of the $H_1(U_j, R)$ in $H_1(M, R)$ therefore spans a subspace of dimension at most $2\nu - 2$, i.e. of codimension at least 2. So we can ensure that all g_i's meet γ by requiring that they define homology classes outside that subspace.

Let U_j be a component of $M - \gamma$. Since $\hat{U}_j$ has geodesic boundary, U_j satisfies the following convexity property: any path in U_j is homotopic in U_j (rel. endpoints) to a unique geodesic path. Also note that every component of $U_j - G$ is simply connected, because it is contained in a pair of pants whose three boundary components belong to G and therefore meet γ.

We are going to construct in U_j a connected 1-complex K_j. This complex has finitely many edges and vertices, but it is not compact (unless $\hat{U}_j$ is compact): in each cusp of U_j there is one edge of K_j going off to infinity. We require that K_j satisfies the following conditions: no vertex belongs to G, each edge is geodesic and meets G, and most importantly each component of $G \cap U_j$ meets K_j exactly once (see Fig. 5 for an example with U_j simply connected).

To construct K_j, first consider a cusp c of U_j bounded by two geodesics g_c and g'_c of γ. There are infinitely many components of $U_j \cap G$ which join g_c to g'_c and bound in U_j a disc containing c; let λ_c be the extreme one (see Fig. 5). Let $V_j \subset \hat{U}_j$ be the compact surface obtained by truncating each cusp c along λ_c. We define a finite set $\mathcal{V}$ by choosing one point in each component of $V_j - G$ (these points will be the vertices of K_j).

Now consider the set $\mathcal{C}$ of geodesic arcs joining two points of $\mathcal{V}$ (possibly equal) and meeting G exactly once. By the convexity of U_j (see above), each component of

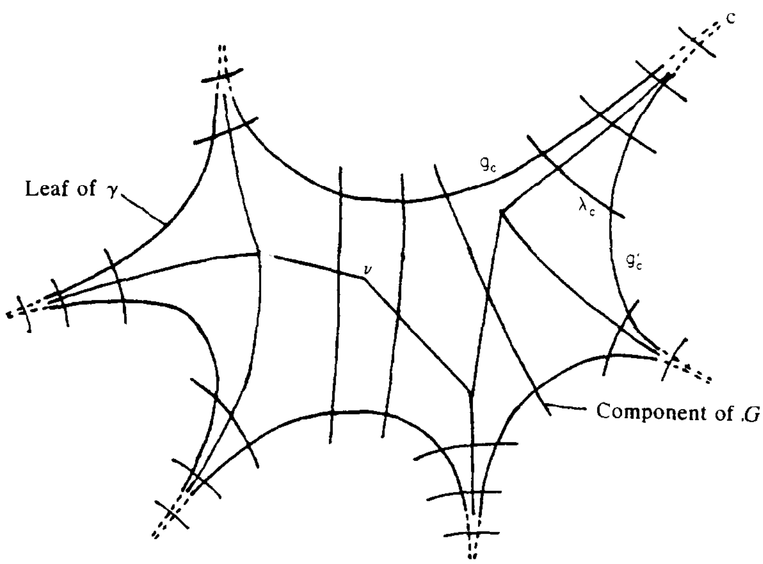


Fig. 5.

$V_j \cap G$ which is not a λ_c meets at least one such arc. Recall that each component of $U_j - G$ is simply connected; it follows that two arcs in $\mathscr{C}$ cannot intersect outside of $\mathscr{V}$, and that any component of $V_j \cap G$ meets at most one arc in $\mathscr{C}$. Elements in $\mathscr{V}$ (resp. $\mathscr{C}$) are therefore the vertices (resp. edges) of a compact connected 1-complex, and we obtain K_j by adding to it the half-geodesics joining a cusp c to the vertex in the component of $V_j - G$ adherent to λ_c.

Remark. There is a proper retraction from $\hat{U}_j$ to K_j. Furthermore K_j can be completed into a foliation $\mathscr{F}_j$ of U_j tangent to the boundary and transverse to G, whose singularities are vertices of K_j. If $\mathscr{F}_j$ has "2-prong saddles", like v on Fig. 5, we get rid of them by a small modification; the foliations $\mathscr{F}_j$ and the lamination γ then fit together into a foliation $\mathscr{F}_m$ of M which satisfies $\gamma(\mathscr{F}_m) = \gamma$. The thick leaves of $\gamma(\mathscr{F}_m)$ are precisely the geodesics isolated on one side. One passes from $\mathscr{F}_m$ to the desired $\mathscr{F}$ by opening up leaves f of $\mathscr{F}_m$ for which $\gamma(f)$ belongs to $e_0 = e - e(\mathscr{F}_m)$, and by collapsing onto K_j components of $U_j - K_j$ bounded by geodesics in $e(\mathscr{F}_m) - e$.

Let P be one of the pairs of pants bounded by G. The surface P contains one or two vertices of the complexes K_j (we do not count vertices adherent to only two edges like v on Fig. 5). The possible patterns for the edges issued from these vertices are pictured on Fig. 6 (up to a permutation of the components of δP). These edges divide δP into 4 or 6 open intervals (I_k, I'_k), with $1 \leq k \leq 2$ or 3, and there exist homeomorphisms u_k: $I_k \to I'_k$ such that, if $p \in I_k$ belongs to a geodesic $g \in \gamma$, then the point $u_k(p) \in I'_k$ is the other endpoint of the arc of $g \cap P$ containing p. Endpoints of intervals I_k or I'_k will be called *base points* on the g_i to which they belong.

Let $]p, q[$ be a component of $g_i - \gamma$; it contains exactly one point k belonging to a complex K_j. Three cases are possible: if neither p nor q belongs to e, we want to collapse $]p, q[$ to a point; if p and q both belong to e, we do not need to do any collapsing; if p (resp. q) belongs to e but q (resp. p) does not, we want to collapse $]k, q[$ (resp. $]p, k[$) to a point.

So we consider on g_i all the open intervals $]a, b[$ disjoint from γ, such that either a and b belong to $\gamma - e$ or a (resp. b) belongs to $\gamma - e$ and b (resp. a) belongs to some K_j. The complement of the union of these intervals is closed and has no isolated point (because every isolated geodesic in γ is thick); therefore we can collapse each of

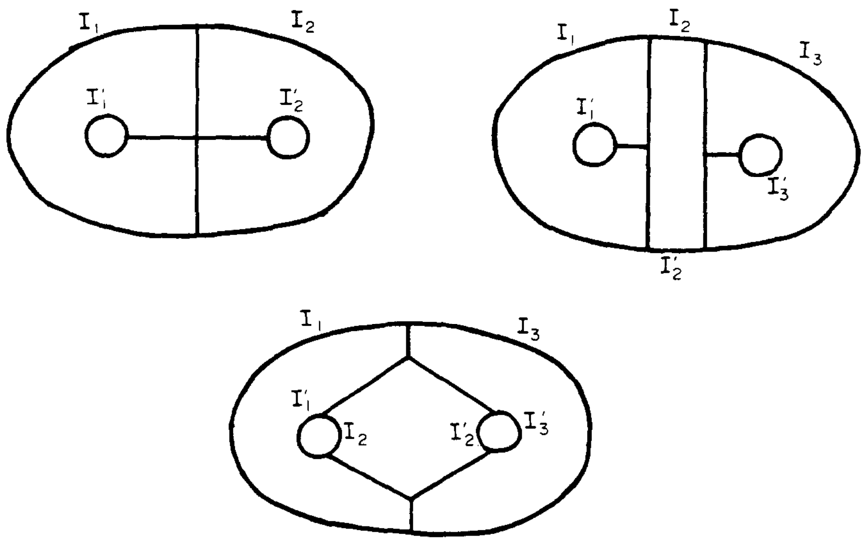


Fig. 6.

these intervals to a point. More precisely, there is a continuous order-preserving map x_i: $g_i \to g_i$ such that $x_i(p_1) = x_i(p_2)$ if and only if p_1 and p_2 belong to an interval $[a, b]$.

We now consider e_0, the set of geodesics in e that are not isolated on any side in γ; note that $\gamma - e_0$ is dense in γ and that the restriction of x_i to $g_i \cap e_0$ is injective. In order to open up points of $x_i(g_i \cap e_0)$, we orient g_i and choose a map y_i: $g_i \to g_i$ which is injective, increasing, right-continuous, and whose points of discontinuity are exactly the points of $x_i(g_i \cap e_0)$. We then define $z_i = \rho_i \circ y_i \circ x_i$, where ρ_i is an orientation-preserving homeomorphism of g_i chosen so that the base points of g_i are fixed points of z_i (such a ρ_i exists because $y_i \circ x_i$ is injective on the set of base points). Note that each I_k or I'_k contained in g_i is sent into itself by z_i.

Let P be a pair of pants bounded by G, and u_k: $I_k \to I'_k$ as above. The interval I_k (resp. I'_k) is contained in a geodesic g_i (resp. $g_{i'}$), and there exists a homeomorphism v_k: $I_k \to I'_k$ such that $v_k(z_i(p)) = z_{i'}(u_k(p))$ for $p \in I_k - e_0$ (equality also holds for $p \in e_0$, provided we replace $z_i(p)$ by the left-limit of z_i at p if necessary).

Now there exists on P a foliation $\mathscr{F}_P$ transverse to the boundary such that, if one of the homeomorphisms v_k is defined at a point $p \in \delta P$, then $v_k(p)$ belongs to the same leaf as p; we also require that the 4 or 6 edges used to define the intervals I_k and I'_k (see Fig. 6) are leaves of $\mathscr{F}_P$. Once the v_k's are chosen, these conditions determine $\mathscr{F}_P$ up to conjugation by a homeomorphism isotopic to the identity rel. δP.

The foliations $\mathscr{F}_P$ fit together into a foliation $\mathscr{F}'$ on M, which by construction satisfies $\gamma(\mathscr{F}') = \gamma$ and $e(\mathscr{F}') = e$. Since every leaf meets G, this foliation has no Reeb component. But, for an arbitrary choice of the homeomorphisms v_k, there is no reason why $\mathscr{F}'$ should satisfy condition (*) if e contains compact geodesics. So we consider, for each compact geodesic $g \in e$, the two extreme leaves f^0 and f^1 such that $\gamma(f^0) = \gamma(f^1) = g$. We modify $\mathscr{F}'$ in the annulus bounded by f^0 and f^1, so as to make all leaves compact; this does not change $\gamma(\mathscr{F}')$ or $e(\mathscr{F}')$. After performing this operation for each compact g in e (there are only finitely many of them), we finally get the desired foliation $\mathscr{F}$. □

§7. PROOF OF 1.2: UNIQUENESS OF $\mathscr{F}$

Let γ, e, and geodesics g_i be as in §6. We want to prove that, up to equivalence, there is only one foliation $\mathscr{F}$ with $\gamma(\mathscr{F}) = \gamma$ and $e(\mathscr{F}) = e$.

LEMMA 3. *Any foliation $\mathscr{F}$ such that $\gamma(\mathscr{F}) = \gamma$ is equivalent to a foliation transverse to the curves g_i.*

Proof of Lemma 3. We use an induction argument. Suppose we have found a foliation $\mathscr{F}_p$ equivalent to $\mathscr{F}$ and transverse to g_i for $1 \le i \le p$ (p may be equal to 0, in which case $\mathscr{F}_0 = \mathscr{F}$). We shall show how to construct $\mathscr{F}_{p+1}$; then $\mathscr{F}_{3\nu-3}$ will be the desired foliation. Let N be the component of $M - (g_1 \cup \cdots \cup g_p)$ which contains g_{p+1}.

Assume first that $\mathscr{F}$ (and therefore also $\mathscr{F}_p$) is a measured foliation. Then it is proved in ([4], Proposition II.6 p. 81, exposé 5) that $\mathscr{F}_p$ can be modified in N in order to give an equivalent foliation $\mathscr{F}_{p+1}$ for which g_{p+1} is transverse or is a leaf cycle or collapses onto a union of saddle connections. But in the latter two cases g_{p+1} either is a leaf of $\gamma(\mathscr{F})$ or is disjoint from $\gamma(\mathscr{F})$, contradicting the definition of the g_i's (see the beginning of §6).

The foliation $\mathscr{F}_{p+1}$ is therefore transverse to g_{p+1}, and also to g_i for i between 1 and p, since $\mathscr{F}_p$ and $\mathscr{F}_{p+1}$ differ only on N. In other words, we see that, to prove Lemma 3, it now suffices to extend Proposition II.6 of [4] to a general foliation.

In the proof of that proposition, the existence of the transverse measure is used only to prove the "lemme de stabilité" (II.4, p. 80). We shall therefore complete the proof of Lemma 3 by showing in §8 that the lemme de stabilité (or "extension lemma") can be applied to any foliation with no Reeb component. □

By Lemma 3, all we need to show is that two foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ with $\gamma(\mathscr{F}_1) = \gamma(\mathscr{F}_2) = \gamma$ and $e(\mathscr{F}_1) = e(\mathscr{F}_2) = e$ are equivalent, provided they are both transverse to G. We can assume without loss of generality that all saddle connections of $\mathscr{F}_1$ and $\mathscr{F}_2$ meet G; then we shall prove that $\mathscr{F}_1$ and $\mathscr{F}_2$ are actually isotopic.

For i between 1 and $3\nu - 3$, choose a component $\tilde{g}_i$ of $p^{-1}(g_i)$. We shall presently construct a map $\tilde{\alpha}_i$ from $\tilde{g}_i$ to itself. First assume $e = \emptyset$. Then there is a bijective correspondence between leaves of $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) and leaves of γ. If $x \in \tilde{g}_i$ belongs to a non-singular leaf h_1 of $\tilde{\mathscr{F}}_1$, let $\tilde{\alpha}_i(x)$ be the point where $\tilde{g}_i$ meets the leaf h_2 of $\tilde{\mathscr{F}}_2$ such that $\gamma(h_2) = \gamma(h_1)$. If x belongs to a separatrix of $\tilde{\mathscr{F}}_1$, then it belongs to exactly two leaves h_1 and h_1'. Let h_2 and h_2' be the leaves of $\tilde{\mathscr{F}}_2$ such that $\gamma(h_2) = \gamma(h_1)$ and $\gamma(h_2') = \gamma(h_1')$. No leaf of $p^{-1}(\gamma)$ meets $\tilde{g}_i$ between $\tilde{g}_i \cap \gamma(h_1)$ and $\tilde{g}_i \cap \gamma(h_1')$; therefore h_2 and h_2' meet $\tilde{g}_i$ at the same point, which we call $\tilde{\alpha}_i(x)$.

Now suppose e is not empty. Let g be in e, and let $\tilde{g}$ be a component of $p^{-1}(g)$. Leaves h of $\tilde{\mathscr{F}}_1$ such that $\gamma(h) = \tilde{g}$ are those located between two extreme leaves h_1^0 and h_1^1 (see §3); we can parametrize them by a number $d_1(h)$ between 0 and 1. We now want to extend the definition of d_1 to all leaves h_1 such that $\gamma(h_1) \in p^{-1}(g)$, in an equivariant way; if $\sigma \in \pi_1 M$, then $d_1(\sigma(h_1)) = d_1(h_1)$. If g is not compact, this is possible because no non-trivial σ sends $\tilde{g}$ to itself. If g is compact, we have $d_1(\sigma(h)) = d_1(h)$ for transformations σ sending $\tilde{g}$ to itself and leaves h with $\gamma(h) = \tilde{g}$, because $\mathscr{F}_1$ satisfies condition (*); the extension is therefore possible. Similarly, define $d_2(h_2)$ for leaves of $\tilde{\mathscr{F}}_2$ such that $\gamma(h_2) \in p^{-1}(g)$, being careful to number h_2^0 and h_2^1 so that h_2^0 is located with respect to h_2^1 on the same side as h_1^0 with respect to h_1^1.

Since g was arbitrary in e, we can attach a number $d_k(h_k)$ to any leaf h_k of $\tilde{\mathscr{F}}_k$ such that $\gamma(h_k) \in p^{-1}(e)$. This allows us to define $\tilde{\alpha}_i$ in the following way: if x belongs to a leaf h_1 of $\tilde{\mathscr{F}}_1$ such that $\gamma(h_1) \notin p^{-1}(e)$, we define $\tilde{\alpha}_i(x)$ as before; if $\gamma(h_1) \in p^{-1}(e)$, we define $\tilde{\alpha}_i(x)$ as the intersection of $\tilde{g}_i$ with the leaf h_2 of $\tilde{\mathscr{F}}_2$ such that $\gamma(h_2) = \gamma(h_1)$ and $d_2(h_2) = d_1(h_1)$. For points x belonging to two leaves of $\tilde{\mathscr{F}}_1$, one checks as in the case $e = \emptyset$ that the two possible definitions of $\tilde{\alpha}_i(x)$ coincide.

The map $\tilde{\alpha}_i$ so defined is an order-preserving bijection from $\tilde{g}_i$ to itself, i.e. a homeomorphism. It commutes with the covering transformations leaving $\tilde{g}_i$ invariant, and therefore induces a homeomorphism α_i on g_i. Note that, because $\tilde{\mathscr{F}}_1$, $\tilde{\mathscr{F}}_2$ and the functions d_k are $\pi_1 M$-equivariant, the map α_i does not depend on the choice of $\tilde{g}_i$ in $p^{-1}(g_i)$. We can extend α_i to a homeomorphism $\bar{\alpha}_i$ of M, equal to the identity outside of a small neighborhood V_i of g_i, isotopic to the identity relatively to $M - V_i$, and whose lifting to $\tilde{M}$ equal to the identity outside of $p^{-1}(V_i)$ induces $\tilde{\alpha}_i$ on $\tilde{g}_i$.

Let $\bar{\alpha} = \bar{\alpha}_1 \circ \bar{\alpha}_2 \circ \cdots \circ \bar{\alpha}_{3\nu-3}$. Then on each pair of pants P_m bounded by G the restrictions of $\mathscr{F}_1$ and $\bar{\alpha}^{-1}(\mathscr{F}_2)$ are conjugate by a homeomorphism of P_m isotopic to the identity relatively to δP_m (because $\mathscr{F}_1$ and $\mathscr{F}_2$ have no saddle connection in P_m). This proves that $\mathscr{F}_1$ and $\mathscr{F}_2$ are isotopic. □

Remark. For measured foliations, one can give a direct proof of the uniqueness of $\mathscr{F}$, using the fact that measured foliations which are measure-equivalent are indeed equivalent ([4, 16]); but proving this fact requires a lot of work, including a lemma analogous to Lemma 3.

§8. THE EXTENSION LEMMA

The question we are considering here is the following: given a segment Δ contained in a leaf of $\mathscr{F}$, how far can Δ be pushed onto neighboring leaves? The similar problem for non-singular codimension 1 foliations of 3-manifolds has been studied by Novikov [9] (for Δ a disc), Roussarie [12] (for Δ an annulus) and Thurston [15] (for Δ any compact surface). The obstruction to pushing Δ is in that case either the existence in Δ of a loop having non-trivial holonomy or the existence of a compact leaf belonging to a "dead-end" component (e.g. a Reeb component).

For foliations of surfaces, the "extension lemma" has been proved in [4] for measured foliations, in [10] for orientable foliations, and in [11] for foliations whose singularities are tripods and thorns, with no connection between singularities and no compact leaf.

Here is a general statement of this lemma:

EXTENSION LEMMA. *Let M be a closed orientable surface, and $\mathscr{F}$ a foliation whose singularities are saddles or thorns. Let $\Delta_0 = [a_0, b_0]$ be a compact interval contained in a leaf of $\mathscr{F}$, and $[a_0, a_1]$, $[b_0, b_1]$ two disjoint intervals transverse to $\mathscr{F}$. Suppose the local holonomy map φ such that $\varphi(a_0) = b_0$ can be extended to a homeomorphism from $[a_0, a_1[$ onto $[b_0, b_1[$, yielding a continuous family of arcs $\Delta_t = [a_t, b_t]$ contained in leaves of $\mathscr{F}$* (for $0 \le t < 1$). *Then one of the following assertions is true*:

(i) *φ can be continuously extended to a_1, yielding an embedded arc Δ_1 joining a_1 to b_1 on a leaf of $\mathscr{F}$.*

(ii) *φ can be extended to a_1, but the limit of the arcs Δ_t as $t \to 1$ contains at least one singularity of $\mathscr{F}$ or is not an embedding or both* (see Fig. 7).

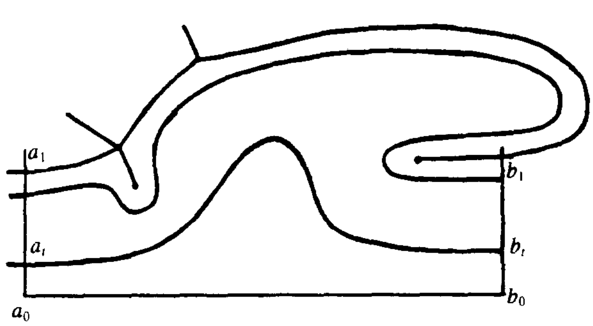


Fig. 7.

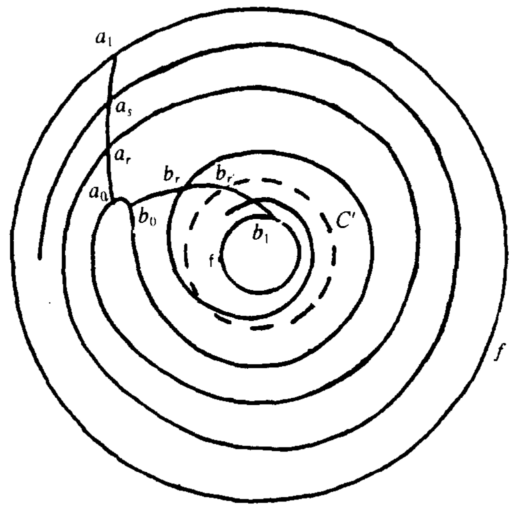


Fig. 8.

(iii) *a_1 and b_1 belong to the boundary of a Reeb component* (see Fig. 8).

Remark. This lemma also applies when M has non-empty boundary, if each component of δM is either transverse to $\mathcal{F}$ or is a leaf.

Proof. (I am grateful to Harold Rosenberg for helping me write a correct proof)

Starting on the same side of a_1 as the arcs Δ_t (the r.h.s. of Fig. 7), follow the leaf containing a_1, turning around if reaching a thorn and making a right turn towards the adjacent separatrix if reaching a saddle. Call f this half-leaf (or union of leaves).

Suppose first that it meets infinitely often some closed transverse curve C. The cardinality $k(t)$ of $\Delta_t \cap C$ is constant near each t for which neither a_t nor b_t belongs to C, and therefore there exists a number K independent of t such that Δ_t meets C at most K times. Choose $x \in f$ such that f meets C at least $K+1$ times between a_1 and x, and fix a transverse interval J containing x. For t close to 1, the segment Δ_t does not meet J, and therefore b_1 has to be on f between a_1 and x, proving that (i) or (ii) holds.

If f spirals towards a compact leaf θ, so do leaves through a_t for t close to 1. Let θ' be a transverse curve close to θ (on the same side of θ as f), and let C be a transverse arc with one endpoint on θ and the other one on $\theta' - f$. For t close to 1, the arc Δ_t meets θ' in at most one point, and this point is close to $f \cap \theta'$; therefore Δ_t cannot contain any endpoint of C, and we can apply the same argument as before.

We now assume that neither (i) nor (ii) holds. Then we know that f (and by symmetry also the leaf f' through b_1) are compact. Note that we can have $f = f'$ only if, as t goes to 1, the points a_t and b_t approach f from opposite sides.

Consider a point a_r, for r close to 1, and follow its leaf f_r towards the left. Since f is compact, one hits $[a_0, a_1]$ again at a point a_s (see Fig. 8). We can find r arbitrarily close to 1 with $s > r$: if not, the supremum of $d(x, f)$ for $x \in \Delta_t$ goes to 0 as t tends to 1 (d is the distance given by some Riemannian metric on M); therefore b_1 belongs to f, $f' = f$, and (i) and (ii) holds.

The curve obtained by following f_r from a_r to a_s, then $[a_0, a_1]$ from a_s to a_r, can be deformed into a closed transverse curve C. Similarly, construct a transverse curve C' close to f', meeting $[b_0, b_1]$ at a point $b_{r'}$ ($r' \geq r$), and disjoint from C.

Leaves meeting C' also meet C and define an isotopy between these two curves;

the annulus A bounded by C and C' is contained in the union of the arcs Δ_t ($s \leq t < 1$). Since neither (i) nor (ii) holds, A is not located between C and f or C' and f', and the union of A with the regions between C and f (resp. C' and f') is the desired Reeb component. Note that its boundary does not necessarily consist of embedded circles if f or f' contains singularities of $\mathcal{F}$. □

REFERENCES

1. S. H. ARANSON and V. Z. GRINES: On some invariants of dynamical systems on two-dimensional manifolds. *Math. USSR Sb.* **19** (1973), 365–393.
2. T. M. CHERRY: Analytic quasi-periodic curves of discontinuous type on a torus. *Proc. London Math. Soc.* **44** (1938), 175–215.
3. A. DENJOY: Sur les courbes définies par les équations différentielles à la surface du tore, *J. Math. Pures et Appl.* **11** (1932), 333–375.
4. A. FATHI, F. LAUDENBACH and V. POENARU: Travaux de Thurston sur les surfaces. *Astérisque* **66–67** (1979), SMF Paris.
5. G. LEVITT: Pantalons et feuilletages des surfaces. *Topology* **21** (1982), 9–33.
6. G. LEVITT: La décomposition dynamique et la différentiabilité des feuilletages des surfaces. in preparation.
7. G. LEVITT: Feuilletages des surfaces, Thèse d'état, in preparation.
8. N. G. MARKLEY: Homeomorphisms of the circle without periodic points. *Proc. London Math. Soc.* **20** (1970), 688–698.
9. S. P. NOVIKOV: Topology of foliations. *Trans. Moscow Math. Soc.* **14**, (1965), 268–304.
10. M. M. PEIXOTO: Structural stability on two-dimensional manifolds. *Topology* **1** (1962), 101–120.
11. H. ROSENBERG: Labyrinths in the disc and surfaces, to appear.
12. R. ROUSSARIE, Plongements dans les variétés feuilletées et classification de feuilletages sans holonomie. *Publ. Math. IHES* **43** (1974), 101–141.
13. A. J. SCHWARTZ: A generalization of a Poincaré–Bendixson theorem to closed two-dimensional manifolds. *Am. J. Math.* **85** (1963), 453–458.
14. P. J. STREDDER: Morse foliations. Thesis. Warwick (1976).
15. W. P. THURSTON: Foliations of 3-manifolds which are circle bundles. Thesis. Berkeley (1972).
16. W. P. THURSTON: On the geometry and dynamics of diffeomorphisms of surfaces, preprint.
17. W. P. THURSTON: The geometry and topology of 3-manifolds. *Notes.* Princeton University.

U.E.R. de mathématiques
Université Paris 7
Paris
France