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CONSTRUCTING FREE ACTIONS ON R-TREES

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Introduction and statement of results. This paper describes two constructions leading to free group actions on **R**-trees. In the first one we start with an arbitrary action (G, T), and we construct actions of certain quotients G/H on quotient **R**-trees $\widehat{T/H}$. Among these actions, there is a "largest" free one, so that we can associate a free action to (G, T) in a canonical way. In the second construction, we use pseudogroups of rotations of the circle constructed in [Le3] to get free nonsimplicial actions of the free group of rank 3. The translation lengths of the generators may be any triple of positive, rationally independent numbers. Both constructions use measured foliations.

To introduce quotient actions, let G be a group acting isometrically on a metric space (X, d), and $H \subset G$ a normal subgroup. Consider X/H, the set of orbits of the restriction of the action to H. The metric of X induces a pseudodistance on X/H, given by $d_H(Hx, Hy) = \inf_{h,h' \in H} d(hx, h'y)$, and we let $\widehat{X/H}$ be the associated metric space. Obviously, G/H acts isometrically on $\widehat{X/H}$.

In order to apply this to **R**-trees, we shall determine when $\widehat{X/H}$ is an **R**-tree, assuming that X is an **R**-tree.

Let therefore H be a group acting on an **R**-tree T. Let $\ell\colon H \to \mathbf{R}^+$ be the associated length function $\ell(h) = \inf_{x \in T} d(x, hx)$. Recall that this infimum is always achieved; in particular, $\ell(h) = 0$ if and only if h acts with a fixed point. (We then say h is *elliptic*.) See the surveys [Sh1], [Sh2], [Mo] for basic facts about **R**-trees.

Given $c \in \mathbf{R}$, with $0 < c \leqslant 1/3$, say that the action of H (or the length function ℓ) satisfies condition $(*)$ if the following holds: given $h \in H$ and $\varepsilon > 0$, one can write $h = h_1 h_2$ with $h_1, h_2 \in H$ and

$$\begin{cases} \ell(h_1) + \ell(h_2) < \ell(h) + \varepsilon \\ \max(\ell(h_1), \ell(h_2)) < (1-c)\ell(h) + \varepsilon. \end{cases} \tag{$*$}$$

THEOREM 1. *Let H be a countable group acting on an* **R**-*tree T, with length function ℓ. Then $\widehat{T/H}$ is an* **R**-*tree if and only if ℓ satisfies* $(*)$.

Remarks. The choice of c in $(0, \frac{1}{3}]$ is irrelevant, but the theorem would be false with $c > \frac{1}{3}$ (see Example III.2). If $\widehat{T/H}$ is isometric to a subinterval of **R**, then ℓ satisfies $(*)$ with $c = \frac{1}{2}$ (see Remark III.1).

If $\ell = |\tau|$, where $\tau\colon H \to \mathbf{R}$ is a homomorphism, then ℓ satisfies $(*)$ if and only if $\tau(H) \not\simeq \mathbf{Z}$.

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We shall see that condition $(*)$ is always satisfied if H is generated by its elliptic elements.

Here and in Theorem 2 below, the countability hypothesis may be removed.

COROLLARY. *Let G be a countable group acting on an* $\mathbf{R}$*-tree T, with length function ℓ. Let $H \subset G$ be a normal subgroup. If the restriction of the action to H satisfies $(*)$, then $\bar{\ell}(gH) = \inf_{h \in H} \ell(gh)$ is a length function on G/H.* ■

The action of G/H on the $\mathbf{R}$-tree $\widehat{T/H}$ is free if and only if the condition

$$g \notin H \Rightarrow \inf_{h \in H} \ell(gh) > 0 \tag{$**$}$$

is satisfied.

Given an action of G, with length function ℓ, it is easy to check that there is a smallest subgroup $H_0 \subset G$ satisfying $(**)$. This subgroup is normal, and it contains all elliptic elements of G. It turns out that $\widehat{T/H_0}$ is an $\mathbf{R}$-tree.

THEOREM 2. *Let G be a countable group acting on an* $\mathbf{R}$*-tree T, with length function ℓ. Let $H_0 \subset G$ be the smallest subgroup satisfying condition $(**)$ above.*

(1) *The space $\widehat{T/H_0}$ is an* $\mathbf{R}$*-tree, and the action of G/H_0 on $\widehat{T/H_0}$ is free.*

(2) *This free action is as big as possible: if $K \subset G$ is a normal subgroup such that G/K acts freely on some* $\mathbf{R}$*-tree with a length function ℓ_K satisfying $\ell_K(gK) \leqslant \ell(g)$ for all $g \in G$, then K contains H_0.*

COROLLARY. *If G is generated by its elliptic elements, then $\widehat{T/G}$ is an* $\mathbf{R}$*-tree.* ■

Here are a few examples.

Example 1. If G acts freely, then of course $H_0 = \{1\}$ and $\bar{\ell} = \ell$.

Example 2. If ℓ is the absolute value of a homomorphism $\tau: G \to \mathbf{R}$, then $H_0 = \ker \tau$ and $\bar{\ell}(gH_0) = \ell(g)$.

Example 3. Geometric examples are provided by measured foliations (see Theorem 7 and Theorem III.7).

Example 4. If G acts simplicially, then H_0 is the subgroup generated by the elliptic elements (since $\ell(G) \subset \mathbf{R}$ is discrete). Bass-Serre theory [Se] implies that G may be reconstructed from a graph of groups, i.e. a graph Δ $(= T/G)$ with certain groups and monomorphisms attached on vertices and edges. The free action given by Theorem 2 is then isomorphic to the action of the fundamental group $\pi_1 \Delta$ (in the usual, topological sense) on the universal covering $\tilde{\Delta}$. In particular, $G/H_0 \simeq \pi_1 \Delta$ is a free group.

Example 5. The group H_0 may be strictly bigger than the subgroup generated by all elliptic elements, even if G is finitely generated. Open up all 3 thorns of the foliation of $\mathbf{P}^2$ described in [AY], so as to get a foliation of a compact nonorientable surface Σ with 3 boundary components. For the associated action of $\pi_1 \Sigma$, it follows from [FLP, prop. II.6 p. 81] that the subgroup generated by the elliptic elements has index 2.

PROPOSITION 3. *Let G be a countable group acting on an* $\mathbf{R}$*-tree T. If the action is simplicial, the following are equivalent*:

(1) *G is generated by its elliptic elements.*

(2) *The free action associated to (G, T) is trivial (i.e. $H_0 = G$).*

(3) *$\widehat{T/G}$ is an* $\mathbf{R}$*-tree.*

(4) *If $\ell \geqslant a|\tau|$, with $a > 0$ and $\tau\colon G \to \mathbf{Z}$ a homomorphism, then $\tau = 0$.*

For arbitrary actions, one simply has $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$.

When G is finitely generated, one can go much further; in particular, (3) and (4) are equivalent [Le4]. For now, let us just consider $PL(G)$, the space of projectivized length functions on G. It is known to be compact [CuMo]. Theorem 2 associates to any $\ell \in PL(G)$ a group $G(\ell) = G/H_0$. Since $G(\ell)$ is finitely generated and acts freely on an $\mathbf{R}$-tree, it is a free product of free abelian groups and surface groups by Rips's theorem (see [GLP], [BF]). Now one may ask the following questions:

(1) Consider the set of $\ell \in PL(G)$ such that $G(\ell)$ is free. Is it dense in $PL(G)$?

(2) Consider the set of $\ell \in PL(G)$ such that $G(\ell)$ is a free product of free abelian groups. Does it contain a dense G_δ in $PL(G)$?

The answer to question 1 is positive if simplicial actions are dense in $PL(G)$, e.g. if G is a free group or a surface group [Sk]. A positive answer to question 2 would mean that the Lyndon conjecture ([Ly], [Ch]) is "true generically" for finitely generated groups.

The "only if" direction in Theorem 1 is easy (see Section III). To prove the other results, we use codimension-one measured foliations $\mathscr{F}$ on open manifolds M.

We shall associate to such an $\mathscr{F}$ a metric space $\hat{T}(\mathscr{F})$, the *leaf space, made Hausdorff*, and give a criterion to decide whether $\hat{T}(\mathscr{F})$ is an $\mathbf{R}$-tree (Theorem III.3).

Using the measure, we can define the *$\mathscr{F}$-length* $\|g\|_{\mathscr{F}}$ of any $g \in \pi_1 M$, as the infimum of the measures of closed curves representing g as a free homotopy class.

The connection between actions on $\mathbf{R}$-trees and measured foliations now comes from the following result. Say that $x \in T$ is an *endpoint* if x does not belong to the interior of a nondegenerate segment. Note that T has no endpoint if some group acts on T minimally (there is no invariant subtree).

THEOREM 4. *Let T be a separable* $\mathbf{R}$*-tree whose set of endpoints is countable. Let H be a countable group acting on T, with length function ℓ. For any $n \geqslant 3$, there exists a (nonsingular) measured foliation $\mathscr{F}$ on an n-dimensional manifold M and an epimorphism $\rho\colon \pi_1 M \twoheadrightarrow H$, such that*:

(1) *$\widehat{T/H}$ is isometric to $\hat{T}(\mathscr{F})$;*

(2) *T is H-equivariantly isometric to $\hat{T}(\mathscr{F}_H)$, where $\mathscr{F}_H$ is the pullback of $\mathscr{F}$ to the normal covering M_H of M defined by ρ;*

(3) $\ker \rho$ *is generated by elements of $\mathscr{F}$-length* 0;

(4) $\ell(h) = \inf_{t \in \rho^{-1}(h)} \|t\|_{\mathscr{F}}$ *for all $h \in H$.*

In particular (see Remark II.4 for a stronger statement), we have the following corollary.

COROLLARY. *Given T as in Theorem 4, there is a measured foliation $(M, \mathscr{F})$ such that $\hat{T}(\mathscr{F})$ is isometric to T and $\pi_1 M$ is generated by elements of $\mathscr{F}$-length 0.* ■

Our second construction will show the following.

THEOREM 5. *Let F_3 be the free group of rank 3. Given three positive, rationally independent numbers α, β, γ, there exists a free nonsimplicial action of F_3 on an $\mathbf{R}$-tree such that α, β, γ are the translation lengths of the generators.*

Remarks. It is known [Ha] that every (minimal) free action of F_2 is topologically conjugate to a simplicial action (polyhedral in the sense of [Sh2]). The actions we obtain are not polyhedral because all orbits are dense. Our construction can be generalized to give uncountably many nonpolyhedral actions of free groups of arbitrary odd rank.

Using automorphisms of free groups, Bestvina and Handel have constructed a countable family of nonsimplicial free actions of free groups (see [Sh2]). The length function associated to any of their actions takes its values in a finite algebraic extension of $\mathbf{Q}$.

To prove Theorem 5, we use measured foliations $\mathscr{F}$ with Morse singularities on closed manifolds M with $\dim M \geqslant 3$ (see Section I). Given $(M, \mathscr{F})$, let $\mathscr{L}$ be the normal subgroup of $\pi_1 M$ generated by those free homotopy classes that can be represented by loops contained in leaves and having trivial holonomy. (We shall distinguish between "loops contained in leaves", which may not pass through singularities, and "loops tangent to $\mathscr{F}$", which may. The quotient $\pi_1 M/\mathscr{L}$ is the fundamental group of Haefliger's classifying space for $\mathscr{F}$; see [Le2, pp. 721–722].)

Let $M(\mathscr{L})$ be the covering corresponding to $\mathscr{L}$. The pullback $\mathscr{F}(\mathscr{L})$ of $\mathscr{F}$ to $M(\mathscr{L})$ can be defined by a Morse function $f: M_{\mathscr{L}} \to \mathbf{R}$. We shall show the following theorem. (See Theorem IV.1 for a more detailed statement.)

THEOREM 6. *Let $(M, \mathscr{F})$ be a Morse measured foliation on a closed manifold. The leaf space of $\mathscr{F}(\mathscr{L})$, made Hausdorff, is an $\mathbf{R}$-tree $\hat{T}(\mathscr{L})$ whose points are in one-to-one correspondence with connected components of level sets $f^{-1}(c)$.*

Now let $\tilde{\mathscr{F}}$ be the pullback of $\mathscr{F}$ to the universal covering $\tilde{M}$, and $\hat{T}(\tilde{\mathscr{F}})$ its leaf space, made Hausdorff. It is known that $\hat{T}(\tilde{\mathscr{F}})$ is an $\mathbf{R}$-tree [GS], but no similar description of its points is known in general (see Section IV).

Theorem 6 readily implies the following corollary.

COROLLARY. *If every loop tangent to $\mathscr{F}$ is null-homotopic in M, then the action of $\pi_1 M$ on the $\mathbf{R}$-tree $\hat{T}(\tilde{\mathscr{F}})$ is free.* ■

Remark. We shall be concerned primarily with the case $\dim M \geqslant 3$, but similar (indeed simpler) techniques apply to measured foliations on surfaces in the sense of Thurston ([Th], [FLP]). One can thus prove that most surface groups act freely on **R**-trees (a result of [MS]), using the point of view of foliations rather than laminations. Note that measured foliations with all leaves dense can be obtained by taking branched coverings of the foliation described in [AY], or (on orientable surfaces) by suspending irreducible irrational interval exchange transformations (in the sense of [Ke]).

To get a free nonsimplicial action of F_3 on an **R**-tree, it now suffices to construct a Morse measured foliation $(M, \mathscr{F})$ such that M is a closed manifold with $\pi_1 M \simeq F_3$, every loop tangent to $\mathscr{F}$ is null-homotopic, and every leaf of $\mathscr{F}$ is dense. (This guarantees that the resulting free action has all its orbits dense; in particular, it is not simplicial.) This is obtained by simply "suspending" pseudogroups of rotations constructed in [Le3, théorème 1].

More generally, any pseudogroup of rotations Γ on the circle gives rise to a free action of a group $G(\Gamma)$. Theorem 7 will lead to a presentation of $G(\Gamma)$ by generators and relations (see Remark IV.4).

Let us return to the situation of Theorem 6. Let $\bar{\mathscr{L}}$ be the normal subgroup of $\pi_1 M$ generated by the free homotopy classes that contain loops tangent to $\mathscr{F}$ (possibly passing through singularities). The above corollary dealt with the case when $\bar{\mathscr{L}} = \{1\}$. In general, the leaf space (made Hausdorff) on the covering corresponding to $\bar{\mathscr{L}}$ is an **R**-tree $\hat{T}(\bar{\mathscr{L}})$, and we shall show the following theorem.

THEOREM 7. *Let $(M, \mathscr{F})$ be a Morse measured foliation on a closed manifold. If $\mathscr{F}$ is transversely orientable (i.e. if $\mathscr{F}$ can be defined by a closed differential 1-form), then $\pi_1 M/\bar{\mathscr{L}}$ acts freely on the* **R**-*tree $\hat{T}(\bar{\mathscr{L}})$.*

This action is the action associated to $(\pi_1 M, \hat{T}(\tilde{\mathscr{F}}))$ by Theorem 2. Note that $\pi_1 M/\bar{\mathscr{L}}$ is a free product of free abelian groups and surface groups (Rips), while $\pi_1 M/\mathscr{L}$ is a free product of free abelian groups [Le2, théorème 1]. There are examples of nonorientable foliations for which Theorem 7 fails.

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I. Preliminaries on measured foliations. Let M^n be an n-dimensional smooth manifold without boundary. A *Morse measured foliation* $\mathscr{F}$ on M is a codimension-one foliation with Morse singularities, equipped with a smooth, holonomy-invariant, transverse measure with full support. Equivalently, the pullback $\tilde{\mathscr{F}}$ of $\mathscr{F}$ to the universal covering $\tilde{M}$ is defined by a Morse function $\tilde{f}$, deck transformations acting by $\tilde{f} \circ \tau = \varepsilon(\tau)\tilde{f} + a(\tau)$, with $\varepsilon(\tau) = \pm 1$ and $a(\tau) \in \mathbf{R}$. A *leaf* of $\mathscr{F}$ will always be a leaf of the nonsingular foliation $\mathscr{F}^*$ induced on $M^* = M - \operatorname{Sing} \mathscr{F}$.

Given $(M, \mathscr{F})$, any path $\gamma \subset M$ has an $\mathscr{F}$-length $|\gamma|_{\mathscr{F}}$, defined as the total mass of the measure induced on γ. For $g \in \pi_1 M$, let $\|g\|_{\mathscr{F}}$ be the infimum of the $\mathscr{F}$-lengths of all closed curves representing g as a free homotopy class; of course, $\|g\|_{\mathscr{F}}$ depends only on the conjugacy class of g. If no confusion is possible, we shall drop the reference to $\mathscr{F}$.

We define a metric space $\hat{T}(\mathscr{F})$ as follows (*leaf space of $\mathscr{F}$ made Hausdorff*). Given x, y in M, let $d_{\mathscr{F}}(x, y)$ be the infimum of the $\mathscr{F}$-lengths of paths from x to y. This defines a pseudodistance on M. (We may have $d_{\mathscr{F}}(x, y) = 0$ for $x \neq y$, for instance, if x and y are on the same leaf.) Identifying points at distance 0 from each other, we get the required metric space $\hat{T}(\mathscr{F})$. In particular, $\hat{T}(\tilde{\mathscr{F}})$ is always an $\mathbf{R}$-tree [GS]. Note that $\pi_1 M$ acts on $\hat{T}(\tilde{\mathscr{F}})$, with length function $g \mapsto \|g\|_{\mathscr{F}}$.

Allowing singularities will be crucial for the proof of Theorem 5 since we want to work on closed manifolds. Recall that very few closed manifolds support nonsingular measured foliations: M (or its 2-sheeted covering making $\mathscr{F}$ transversely orientable) must fiber over S^1 [Ti].

Let s be a critical point of a Morse function g. If s has index different from 1 or $n-1$, there exist arbitrarily small neighborhoods U_s such that all level sets of $g|U_s$ are connected. This is not true if s has index 1 or $n-1$. Assuming $n \geqslant 3$, the level set of s then consists of $\{s\}$, and of 2 *singular half-cones* which get separated on nearby levels on one side (see [Le1, I.4]). This explains why the important singularities of $\mathscr{F}$ will be those of index 1 or $n-1$. (Note that singularities of $\mathscr{F}$ have an index, well-defined up to replacing p by $n-p$.)

II. From trees to foliations. This section is devoted to the proof of Theorem 4. We are given a separable $\mathbf{R}$-tree T with countably many endpoints, equipped with an action of a group H. We are looking for a foliation $\mathscr{F}$ with $\hat{T}(\mathscr{F})$ isometric to $\widehat{T/H}$.

We shall not treat the (not so trivial, but useless) case when T is a point. The assumptions made on T guarantee that T may be covered by countably many segments (cf. [MNO, Prop. 1.6]). More precisely, we have the following lemma.

LEMMA II.1. *Let T be as in Theorem 4. Assume T is not a point. There is a bi-infinite sequence of real numbers $\cdots < a_{-2} < a_{-1} < a_0 < a_1 < \cdots$ with $\lim_{|n| \to +\infty} |a_n| = +\infty$, and a continuous surjective map $q: \mathbf{R} \to T$, with the following properties*:

(1) *for all i, the restriction of q to $A_i = [a_i, a_{i+1}]$ is an isometry onto some segment $B_i \subset T$;*

(2) *for all i, j, the intersection $B_i \cap B_j$ is either empty or a nondegenerate segment (but not a point).*

Proof. Set $a_0 = 0$. We define q on $\mathbf{R}^+$ and then extend it by $q(-t) = q(t)$. Let $(x_n)_{n \in N}$ be a dense subset of T containing all endpoints. Define a_n and q by requiring that q map $[a_n, a_{n+1}]$ isometrically onto the segment $[x_n x_{n+1}] \subset T$. We get a surjective map $q: \mathbf{R} \to T$ that satisfies condition (1).

It is then easy to modify q so that it also satisfies (2). This is done inductively, in such a way that the finite subtree $T_n = q([a_0, a_n])$ has the following property: for every vertex v of T_n, there is an edge $e \ni v$ such that every B_i $(0 \leqslant i \leqslant n-1)$ containing v also contains e. One may have to add subdivision points in the process. We leave details to the reader. ∎

Note that, since q is onto, every segment in T is contained in a union of finitely many B_i's.

Let S be the set of $s = (i, j, h) \in \mathbf{Z} \times \mathbf{Z} \times H$ such that $hB_i \cap B_j$ is a nondegenerate segment; denote $h = \sigma(s)$. Given $s \in S$, let I_s (resp. J_s) be the nondegenerate closed subinterval of A_i (resp. A_j) consisting of all $x \in A_i$ such that $hq(x) \in B_j$ (resp. all $y \in A_j$ such that $q(y) \in hB_i$). Let γ_s: $I_s \to J_s$ be the isometry such that $q \circ \gamma_s = h \circ q$.

We have thus obtained a countable family $(\gamma_s)_{s \in S}$ of partial isometries of $\mathbf{R}$. We now show how to recover H from it. (This will be used for the third assertion of Theorem 4.)

In the free group $F(S)$, take as relations all words $s_1 \cdots s_n$ such that $\sigma(s_1) \cdots \sigma(s_n) = 1$ and there exists $x \in \mathbf{R}$ with $\gamma_{s_1} \cdots \gamma_{s_n}(x)$ defined and equal to x. Let π: $F(S) \to F(S)/N_S$ be the corresponding quotient homomorphism. The homomorphism ρ_S: $F(S) \to H$ sending s to $\sigma(s)$ factors through $\bar{\rho}_S$: $F(S)/N_S \to H$.

LEMMA II.2. *The homomorphism $\bar{\rho}_S$: $F(S)/N_S \to H$ is an isomorphism.*

Proof. Given $i \in \mathbf{Z}$ and $h \in H$, there is a $j \in \mathbf{Z}$ such that $(i, j, h) \in S$. In particular, $\bar{\rho}_S$ is onto. Also note $\pi(i, j, h)\pi(j, i, h^{-1}) = 1$ for all $(i, j, h) \in S$. The injectivity of $\bar{\rho}_S$ is proved in three steps:

1. If $\sigma(s) = 1$, then $\pi(s) = 1$ in $F(S)/N_S$.
2. More generally, $\pi(s)$ depends only on $\sigma(s)$.
3. If $\sigma(s) = \sigma(s_1)\sigma(s_2)$, then $\pi(s) = \pi(s_1)\pi(s_2)$.

Proof of 1. It is by contradiction. Among all $s \in S$ with $\sigma(s) = 1$ and $\pi(s) \neq 1$, choose one for which $i < j$ and $j - i$ is minimal. By minimality, $q([a_{i+1}, a_j])$ is disjoint from $q(I_s) = q(J_s)$; in particular, $q(a_{i+1})$ and $q(a_j)$ are in the same component of $T - q(I_s)$. Let x be the right endpoint of I_s. We know $x \neq a_{i+1}$. Similarly, let $y \neq a_j$ be the left endpoint of J_s. Since $q([x, a_{i+1}]) \cap q([a_j, y]) = q(x) = q(y)$, the points $q(a_{i+1})$ and $q(a_j)$ are in distinct components of $T - q(x)$, a contradiction.

Proof of 2. Let $s = (i, j, h)$ and $s' = (i', j', h)$ be in S. First, assume $i = i'$ and consider the segment hB_i. Choose a finite number of nondegenerate segments $C(j_l) = hB_i \cap B_{j_l}$ $(1 \leqslant l \leqslant p)$, with $j_1 = j$, $j_p = j'$, and $C(j_l) \cap C(j_{l+1}) \neq \varnothing$. Since $\pi(i, j_{l+1}, h) = \pi(j_l, j_{l+1}, 1)\pi(i, j_l, h)$, we get $\pi(i, j, h) = \pi(i, j', h)$. Given i, let j be such that $B_j \cap hB_i$ and $B_j \cap hB_{i+1}$ are nondegenerate segments. Then $\pi(i, j, h) = \pi(i + 1, j, h)$, so that 2 is proved.

Proof of 3. Choose any i. Choose j such that $\sigma(s_2)B_i \cap B_j$ is a nondegenerate segment C. Finally, choose k such that $\sigma(s_1)C \cap B_k$ is a nondegenerate segment. Then $\pi(i, k, \sigma(s_1)\sigma(s_2)) = \pi(j, k, \sigma(s_1))\pi(i, j, \sigma(s_2))$. ■

We shall now associate a measured foliation $\mathscr{F}$ to the system $(\gamma_s)_{s \in S}$. Start with $\mathbf{R}^n = \mathbf{R}^{n-1} \times \mathbf{R}$ $(n \geqslant 3)$, with the measured foliation $\mathscr{F}_0$ given by the projection p: $\mathbf{R}^n \to \mathbf{R}$. Using p, we lift the subintervals I_s, J_s of $\mathbf{R}$ to intervals transverse to $\mathscr{F}_0$. Thanks to the $\mathbf{R}^{n-1}$ factor, we can place all those intervals far away from each other.

Then perform foliated connected sums as in [AL, pp. 144–145]. Consider the simply connected manifold (with boundary) M_1 obtained by removing for every $s \in S$ the interior of a thin ellipsoid $BI(s)$ (resp. $BJ(s)$) having I_s (resp. J_s) as its big axis. Then glue $\delta BI(s)$ to $\delta BJ(s)$ by a diffeomorphism $\theta(s)$ such that $p \circ \theta(s) = \gamma_s \circ p$.

We get a manifold M without boundary, and the foliation $\mathscr{F}_0$ induces a measured foliation $\mathscr{F}$ on M, with Morse singularities of index 1 or $n-1$ coming from the endpoints of the intervals I_s, J_s.

Since $n \geqslant 3$, leaves of $\mathscr{F}$ are in one-to-one correspondence with orbits of H on T (provided we identify leaves containing two singular half-cones issuing from the same singularity). We shall show that $(M, \mathscr{F})$ is indeed the desired foliation. (If one insists on getting a nonsingular foliation, one can simply remove the singularities from M; since $n \geqslant 3$, this will not change the fundamental group.)

Clearly, $\pi_1 M$ is isomorphic to the free group $F(S)$. We fix an explicit identification $\lambda: \pi_1 M \xrightarrow{\simeq} F(S)$ as follows.

Let $\gamma_1, \ldots, \gamma_k$ be paths in M_1, such that the endpoint β_i of γ_i is in some $\delta BJ(s_i)$ $(1 \leqslant i \leqslant k-1)$, the origin α_i of γ_i is in $\delta BI(s_{i-1})$ $(2 \leqslant i \leqslant k)$, and $\beta_i = \theta(s_i)(\alpha_{i+1})$ for $1 \leqslant i \leqslant k-1$. This data represents a path $\gamma = \gamma_1 \cdots \gamma_k$ in M. We shall say that such a γ is a *good* path from α_1 to β_k. Of course, any path in M can be approximated by a good path.

For a good path γ, define $\lambda(\gamma) = s_1 \cdots s_{k-1} \in F(S)$ and $\rho(\gamma) = \sigma(s_1) \cdots \sigma(s_{k-1})$. Restricting to (good) loops, we get an isomorphism $\lambda: \pi_1 M \to F(S)$ and an epimorphism $\rho = \rho_S \circ \lambda: \pi_1 M \to H$. (All this is independent of a choice of basepoint since M_1 is simply connected).

By Lemma II.2, the subgroup $N = \ker \rho \subset \pi_1 M$ is generated by elements of $\mathscr{F}$-length 0, so that assertion 3 of Theorem 4 is proved.

Now let $p_1: M_1 \to \mathbf{R}$ be the restriction of $p: \mathbf{R}^n \to \mathbf{R}$, and $q_1 = q \circ p_1: M_1 \to T$. If γ is a good path, its $\mathscr{F}$-length is the sum of the total variations of p_1 along the γ_i's. Also note the equalities $q_1(\beta_i) = \sigma(s_i) q_1(\alpha_{i+1})$, $1 \leqslant i \leqslant k-1$.

LEMMA II.3. *For $\alpha, \beta \in M_1$ and $h \in H$, we have $d(q_1(\alpha), hq_1(\beta)) = \inf\{|\gamma|_{\mathscr{F}}; \gamma$ good path from α to β such that $\rho(\gamma) = h\}$.*

This lemma easily implies assertions 1 and 4 of Theorem 4. Proving assertion 2 is a little less immediate, but we shall not give details as it is not used in the sequel.

Remark II.4. The proof will show that the infimum is achieved. For the Morse measured foliation induced on the covering M_H, this means that the pseudodistance between two points is always realized by a path.

Proof of Lemma II.3. Let γ be a good path from $\alpha_1 = \alpha$ to $\beta_k = \beta$, with $\rho(\gamma) = h$. Clearly,

$$
\begin{aligned}
|\gamma|_{\mathscr{F}} = \sum_{i=1}^{k} |\gamma_i|_{\mathscr{F}} &\geqslant \sum_{i=1}^{k} |p_1(\alpha_i) - p_1(\beta_i)| \geqslant \sum_{i=1}^{k} d(q_1(\alpha_i), q_1(\beta_i)) \\
&\geqslant \sum_{i=1}^{k} d(\sigma(s_1) \cdots \sigma(s_{i-1}) q_1(\alpha_i), \sigma(s_1) \cdots \sigma(s_{i-1}) q_1(\beta_i)) \\
&\geqslant d(q_1(\alpha_1), hq_1(\beta_k)).
\end{aligned}
$$

Conversely, we find a γ with $|\gamma|_{\mathscr{F}} = d(q_1(\alpha), hq_1(\beta))$. It clearly exists if $q_1(\alpha) = hq_1(\beta)$ (first check the case $h = 1$). In general, subdivide the segment $[q_1(\alpha), hq_1(\beta)]$ by points $x_1 = q_1(\alpha), \ldots, x_{k+1} = hq_1(\beta)$, so that each $[x_i, x_{i+1}]$ is contained in some B_{j_i}. Define $s_i = (j_{i+1}, j_i, 1)$ for $1 \leqslant i \leqslant k-1$ and construct a good path $\gamma' = \gamma_1 \cdots \gamma_k$ such that q_1 maps each γ_i isometrically to $[x_i, x_{i+1}]$. Clearly, $|\gamma'|_{\mathscr{F}} = d(q_1(\alpha), hq_1(\beta))$ and $\rho(\gamma') = 1$. The existence of γ now follows since the endpoints u and v of γ' satisfy $q_1(u) = q_1(\alpha)$ and $q_1(v) = hq_1(\beta)$. ∎

III. Is the leaf space an R-tree? In this section we shall use Theorem 4 to prove Theorems 1 and 2 and Proposition 3, but first we show the "only if" direction in Theorem 1: if $\widehat{T/H}$ is an **R**-tree, then ℓ satisfies $(*)$.

Assume $\widehat{T/H}$ is an **R**-tree and let $\pi: T \to \widehat{T/H}$ be the projection. Fix $h \in H$. There is nothing to prove if $\ell(h) = 0$; so assume $\ell(h) > 0$. Identify the translation axis A_h of h with **R**, so that h acts by $x \mapsto x + \ell(h)$. Consider points $x_i = i\ell(h)/3$, $0 \leqslant i \leqslant 3$. Note that $\pi(x_3) = \pi(x_0)$. Since $\pi([x_i, x_{i+1}])$ contains the segment $[\pi(x_i), \pi(x_{i+1})]$, there exist points y_i $(1 \leqslant i \leqslant 3)$ with $x_{i-1} \leqslant y_i \leqslant x_i$ and $\pi(y_1) = \pi(y_2) = \pi(y_3)$. At least one couple (j, j') will satisfy $\frac{1}{3} \leqslant |y_j - y_{j'}| \leqslant \frac{2}{3}$, and we take for h_1 an element of H sending y_j very close to $y_{j'}$.

Remark III.1. Suppose $\widehat{T/H}$ is isometric to a subinterval of **R**. View π as a real-valued function. By the intermediate-value theorem, the function $\pi(x + \ell(h)/2) - \pi(x)$ vanishes for some $x \in A_h$. This shows that ℓ satisfies $(*)$ for $c = \frac{1}{2}$. The converse holds for minimal free actions of finitely generated groups [Le4].

Example III.2. This example shows that c cannot be taken greater than $\frac{1}{3}$ in Theorem 1. Let Y be the finite simplicial tree having three vertices v_1, v_2, v_3 of valence 1 and one vertex v of valence 3. Make it into a graph of groups by setting $G(v_i) = \mathbf{Z}/2\mathbf{Z}$ and $G(v) = \{1\}$ (all edge groups are trivial). The corresponding action of $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ satisfies $(*)$ if and only if $c \leqslant \frac{1}{3}$.

Now let $\mathscr{F}$ be a (nonsingular) measured foliation on a manifold M without boundary. Let C be a closed curve in M. We shall always assume that C is oriented, and in general position with respect to $\mathscr{F}$: C is transverse to $\mathscr{F}$ except at finitely many points where simple tangencies occur. We let $|C|$ be the $\mathscr{F}$-length of C, and $\|C\|$ the infimum of $|C'|$ for C' freely homotopic to C (i.e. $\|C\| = \|g\|$, if $g \in \pi_1 M$ is represented by C).

Given $p, q \in C$, the orientation of C allows us to distinguish two arcs pq and qp; arcs will always be *closed* arcs. The *distance on* C between p and q is the $\mathscr{F}$-length of the shorter of the two arcs pq and qp. Given four points p_1, p_2, p_3, p_4 in that order on C, we say that two points $x, y \in C$ are *on opposite arcs* if one is on the arc p_1p_2 and the other on p_3p_4, or if one is on p_4p_1 and the other on p_2p_3.

Say that $x, y \in M$ are *$\mathscr{F}$-equivalent* if $d_{\mathscr{F}}(x, y) = 0$.

THEOREM III.3. *Let $\mathscr{F}$ be a measured foliation on a manifold M and $c \in (0, \frac{1}{3}]$. The following conditions are equivalent*:

(1) *$\hat{T}(\mathcal{F})$ is an* **R**-*tree.*
(2) *Given four points on a closed curve, there exist $\mathcal{F}$-equivalent points on opposite arcs.*
(3) *Every closed curve $C \subset M$ contains $\mathcal{F}$-equivalent points whose distance on C is at least $c\|C\|$.*
(4) *Given $g \in \pi_1 M$ and $\varepsilon > 0$, there exist $g_1, g_2 \in \pi_1 M$ such that $g = g_1 g_2$ and*

$$\begin{cases} \|g_1\| + \|g_2\| < \|g\| + \varepsilon \\ \max(\|g_1\|, \|g_2\|) < (1 - c)\|g\| + \varepsilon. \end{cases}$$

Remarks. The implication (2) ⇒ (1) is known: it is used in [Pa, proof of Prop. 4.6] to prove that $\hat{T}(\mathcal{F})$ is an **R**-tree if M is simply connected (a result of [GS]).

Another equivalent condition is that $\hat{T}(\mathcal{F})$ contains no embedded circle.

The rest of this section is organized as follows. First, we prove a geometric result (Lemma III.4), and we deduce Theorem 2 and Proposition 3. Then we prove Theorem III.3, the main steps being (4) ⇒ (3) and (3) ⇒ (2). Finally, we use (4) ⇒ (1) to show Theorem 1.

Replacing M by $M \times \mathbf{R}^p$ with the product foliation if necessary, we may assume that dim M is large, so that we need only consider smoothly *embedded* curves and surfaces.

We shall use (singular) foliations $\mathcal{G}$ induced by $\mathcal{F}$ on compact surfaces $P \subset M$. We always assume general position, so that $\mathcal{G}$ has only Morse singularities (centers and 4-prong saddles) and δP is transverse to $\mathcal{G}$ except at finitely many points. Since $\mathcal{G}$ is a measured foliation, Poincaré recurrence (see [FLP, exposé 5]) implies that all but finitely many leaves leaving δP must return to δP.

LEMMA III.4. *Let $\mathcal{F}$ be a measured foliation on a manifold M. Let $C \subset M$ be a closed curve, and P a compact orientable surface of genus* 0 *whose boundary consists of C and* (*possibly*) *other curves $C_1, \ldots, C_k$. Given four points on C, there exist points $x, y \in C$ on opposite arcs such that $d_{\mathcal{F}}(x, y) \leqslant \frac{1}{4}\sum |C_i|$.*

Remark. This lemma is implicit in [Pa] for P a disk.

Proof of Lemma III.4. Let $\mathcal{G}$ be the foliation induced on P. In this proof, "distance" will always refer to $d_{\mathcal{G}}$. Let p_1, p_2, p_3, p_4 (in that order) be the points given on C. We may assume that the measure of the set of points of the arc $p_1 p_2$ whose leaf reaches $\bigcup C_i$ is at most $\frac{1}{4}\sum |C_i|$ and that no regular leaf of $\mathcal{G}$ joins $p_1 p_2$ to $p_3 p_4$.

Among all points of $p_1 p_2$ whose distance to $p_4 p_1$ (resp. $p_2 p_3$) is 0, let q (resp. r) be the one closest to p_2 (resp. p_1). If r is between p_1 and q, the distance between $p_4 p_1$ and $p_2 p_3$ is 0 (because P has genus 0). If not, a regular leaf meeting qr must either return to qr or reach some C_i. (It cannot reach $p_1 q$ or $r p_2$, again because of genus 0.) It follows that the distance between q and r is at most $\frac{1}{4}\sum |C_i|$, so that $d_{\mathcal{F}}(p_4 p_1, p_2 p_3) \leqslant d_{\mathcal{G}}(p_4 p_1, p_2 p_3) \leqslant \frac{1}{4}\sum |C_i|$. ■

COROLLARY III.5. *Let $\mathscr{F}$ be a measured foliation. Suppose that, given $t \in \pi_1 M$ and $\varepsilon > 0$, one can write $t = t_1 \cdots t_k$ with $\sum \|t_i\| < \varepsilon$. (This happens for instance if $\pi_1 M$ is generated by elements of $\mathscr{F}$-length 0.) Then $\hat{T}(\mathscr{F})$ is an* **R**-*tree.*

Proof. We check condition (2) of Theorem III.3. (Recall that it is known to imply condition (1).) Consider four points on a closed curve C. Given $\varepsilon > 0$, let P be as in Lemma III.4, with $\sum |C_i| < \varepsilon$. By Lemma III.4, there exist points on opposite arcs that are $\varepsilon/4$-close for $d_{\mathscr{F}}$. Letting ε go to 0, an easy compactness argument then yields $\mathscr{F}$-equivalent points. ■

COROLLARY III.6. *Let H be a countable group acting on an* **R**-*tree T with length function ℓ. Suppose that, given $h \in H$ and $\varepsilon > 0$, one can write $h = h_1 \cdots h_p$ with $\sum \ell(h_i) < \varepsilon$. Then $\widehat{T/H}$ is an* **R**-*tree.*

Proof. Recall [AB, Theorem 3.17] that a connected metric space (X, d) is an **R**-tree if and only if, given any four points x_i, the numbers $d(x_i, x_j)$ satisfy a certain inequality (known as 0-hyperbolicity). Replacing T by the smallest H-invariant subtree containing four given points, if necessary, we may therefore assume that T satisfies the hypothesis of Theorem 4.

Apply Theorem 4 to the action of H. We claim that the corresponding foliation $\mathscr{F}$ satisfies the hypothesis of Corollary III.5.

Given $t \in \pi_1 M$ and $\varepsilon > 0$, we can write $\rho(t) = h_1 \cdots h_p$ with $h_i \in H$ and $\sum \ell(h_i) < \varepsilon$. Using assertion (4) of Theorem 4, choose $t_i \in \rho^{-1}(h_i)$ such that $\sum \|t_i\|_{\mathscr{F}} < \varepsilon$. Then define n by $t = t_1 \cdots t_p n$. By assertion (3) of Theorem 4, it is a product of elements of $\mathscr{F}$-length 0, so that t has the required decomposition.

It follows that $\hat{T}(\mathscr{F})$, and hence also $\widehat{T/H}$, is an **R**-tree. ■

Proof of Theorem 2. First, we show the existence of a smallest subgroup H_0 satisfying (**). Say that a subgroup $H \subset G$ is a (**)-subgroup if it satisfies (**). Any intersection of (**)-subgroups is a (**)-subgroup. Any conjugate of a (**)-subgroup is a (**)-subgroup. It follows that H_0 exists and is normal.

Let H' consist of all $h \in H_0$ such that, for every $\varepsilon > 0$, one can write $h = h_1 \cdots h_p$ with $h_i \in H_0$ and $\sum \ell(h_i) < \varepsilon$. One easily checks that H' is a (**)-subgroup, so that $H' = H_0$. Corollary III.6 now implies that $\widehat{T/H_0}$ is an **R**-tree.

Since H_0 satisfies (**), the action of G/H_0 on $\widehat{T/H_0}$ is free.

To prove assertion (2) of Theorem 2, simply note that K is a (**)-subgroup since $g \notin K \Rightarrow 0 < \ell_K(gK) \leqslant \inf_{k \in K} \ell(gk)$. ■

Similar arguments may be applied directly to any measured foliation $(M, \mathscr{F})$. One gets the following description of the free action associated to the action of $\pi_1 M$ on $\hat{T}(\tilde{\mathscr{F}})$.

THEOREM III.7. *Let $\mathscr{F}$ be a measured foliation on a manifold M. Let $H_0 \subset \pi_1 M$ be the smallest subgroup such that $g \notin H_0 \Rightarrow \inf_{h \in H_0} \|gh\|_{\mathscr{F}} > 0$. Then $\pi_1 M/H_0$ acts freely on the* **R**-*tree $\hat{T}(\mathscr{F}_0)$, where $\mathscr{F}_0$ is the pullback of $\mathscr{F}$ to the covering of M corresponding to H_0.* ■

Proof of Proposition 3. We first show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) for arbitrary actions.

(1) $\Rightarrow$ (2) is clear since H_0 contains all elliptic elements.

(2) $\Rightarrow$ (3) follows from Theorem 2 since $\widehat{T/H_0}$ is an $\mathbf{R}$-tree.

(3) $\Rightarrow$ (4) is a consequence of the "only if" direction of Theorem 1: condition (*) prohibits an inequality of the form $\ell \geqslant a|\tau|$, with $a > 0$ and τ a nonzero homomorphism to $\mathbf{Z}$.

Now we assume that G acts simplicially, and we prove (4) $\Rightarrow$ (1). Example 4 (before the statement of Proposition 3) gives an action of $G/H_0 \simeq \pi_1\Delta$ on $\tilde{\Delta}$ by covering transformations. If (1) does not hold, then Δ is not a tree and the length function of this action is bounded below by $|\tau_0|$, with $\tau_0\colon G/H_0 \twoheadrightarrow \mathbf{Z}$ an epimorphism. Composing τ_0 with the projection $G \to G/H_0$, we get an epimorphism $\tau\colon G \twoheadrightarrow \mathbf{Z}$ such that $\ell \geqslant |\tau|$, and (4) does not hold. ■

Remark. For nonsimplicial actions, all three implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) admit counterexamples. Both (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) admit counterexamples with G finitely generated.

Proof of Theorem III.3. First, assume that $\hat{T}(\mathscr{F})$ is an $\mathbf{R}$-tree and consider three points q_1, q_2, q_3 (in that order) on an oriented closed curve C. Let z_i be their images in $\hat{T}(\mathscr{F})$. Since the segments $[z_1z_2]$, $[z_2z_3]$, $[z_3z_1]$ have a nonempty intersection, there exist $\mathscr{F}$-equivalent points p, q, r on q_1q_2, q_2q_3, and q_3q_1. If furthermore q_1, q_2, q_3 are equidistant on C, two of the points p, q, r will have distance at least $|C|/3$ on C.

Using these remarks, it is easy to see that condition (1) implies the others. The proof of Theorem III.3 now proceeds in two steps (recall that (2) $\Rightarrow$ (1) is known). First we show that, if condition (4) holds for some $c > 0$, then condition (3) holds for some $c'(=c/16)$. Then we prove (3) $\Rightarrow$ (2).

Proof of $(4)_c \Rightarrow (3)_{c'}$. By compactness it suffices, given C and $\varepsilon > 0$, to find points, ε-close for $d_{\mathscr{F}}$, whose distance on C is $\geqslant c\|C\|/16$.

Let $g \in \pi_1 M$ be represented by C. We may assume that $\|g\| = \|C\|$ is strictly positive, and we limit ourselves to $\varepsilon < c\|g\|/4$. Choose $\delta > 0$ with $\delta < \varepsilon/4$ and note $\varepsilon + \delta < c\|g\|/3$. Define g_1 and g_2 using condition (4) and represent each g_i by a curve C_i with $|C_i| < \|g_i\| + \delta$. We now consider the foliation $\mathscr{G}$ induced on a pair of pants bounded by $C \cup C_1 \cup C_2$.

We distinguish three cases. First, assume there are regular leaves of $\mathscr{G}$ going from C to C_i $(i = 1, 2)$. Let $I_i = (a_i, b_i)$ be the smallest open interval in C meeting all regular leaves from C to C_i, such that any two such leaves are isotopic as arcs from I_i to C_i. (Of course, $a_1 = b_2$ and/or $b_1 = a_2$ is possible.)

Let $A_1 \in C_1$ be the limit, when $a \in I_1$ tends to a_1, of the other endpoint of the leaf through a. Note that a_1 and A_1 are $\mathscr{G}$-equivalent. (They can be joined by a union of singular leaves, but we do not need this.) Define A_2, B_1, B_2 similarly. Let γ_i be the subarc of C_i between A_i and B_i that does not meet the leaves from C to C_i.

We claim $|\gamma_i| < \varepsilon$. We give the proof for γ_1. A regular leaf of $\mathscr{G}$ meeting γ_1 is either a nonessential arc from γ_1 to γ_1, or an arc from γ_1 to γ_2. These two types of leaves meet γ_1 in sets of measures θ_{11} and θ_{12} respectively, with $|\gamma_1| = \theta_{11} + \theta_{12}$. The inequality $|\gamma_1| < \varepsilon$ now follows from $\theta_{11} \leqslant \delta < \varepsilon/4$ and

$$\|g\| \leqslant |C_1| + |C_2| - 2\theta_{12} \leqslant \|g_1\| + \|g_2\| + 2\delta - 2\theta_{12} \leqslant \|g\| + \varepsilon + 2\delta - 2\theta_{12}.$$

Since a_1 and b_1 are ε-close, we are done unless one of the arcs I_1, I_2 is smaller than $c\|C\|/16$, say $|I_1| < c\|C\|/16$. Using the inequalities

$$\begin{cases} \|g\| \leqslant |b_2 a_1| + |I_1| + |b_1 a_2| + |C_2| \\ |C_2| \leqslant \|g_2\| + \delta \leqslant (1-c)\|g\| + \varepsilon + \delta, \end{cases}$$

we then get $|b_2 a_1| + |b_1 a_2| \geqslant c\|g\|/2$ since $\varepsilon + \delta < c\|g\|/3$.

Assume for instance $|b_1 a_2| \geqslant c\|g\|/4$. Consider a nullhomotopic simple closed curve C_0 consisting of $b_1 a_2$, γ_2, $b_2 a_1$, γ_1, together with very small arcs joining a_2 to A_2, B_2 to b_2, a_1 to A_1, B_1 to b_1. Let p_1, p_2, p_3, p_4 be points on $b_1 a_2$ with $p_1 = b_1$ and $|p_j p_{j+1}| = c\|g\|/16$ $(j = 1, 2, 3)$.

Applying Lemma III.4 to these four points on C_0, with P a disc, we get $\mathscr{F}$-equivalent points on opposite arcs. One of these two points may fail to be in C, but then it is ε-close to b_1 or a_2. This completes the proof in the first case.

The second case is when there are regular leaves from C to C_1, but not from C to C_2 (or vice versa). Define I_1, a_1, b_1, A_1, B_1, γ_1 as before. Note that a leaf going from γ_1 to itself cannot be an essential arc in P since this would lead to

$$\|g\| \leqslant |C_1| \leqslant \|g_1\| + \delta \leqslant (1-c)\|g\| + \varepsilon + \delta,$$

contradicting the assumption $\varepsilon + \delta < c\|g\|/3$. The same argument as before then shows $|\gamma_1| < \varepsilon$ and $|C_2| < \varepsilon$.

Let J_1 be the complement of I_1 in C. Since $\|g\| \leqslant |J_1| + |C_1|$ and $|C_1| \leqslant (1-c)\|g\| + \varepsilon + \delta$, we get $|J_1| \geqslant c\|C\|/2$. On J_1, choose points $p_1 = b_1, p_2, p_3, p_4$ with $|p_j p_{j+1}| \geqslant c\|C\|/8$ and consider C_0 consisting of J_1, γ_1, and small arcs from a_1 to A_1 and from B_1 to b_1. Then apply Lemma III.4, with P the annulus bounded by C_0 and C_2. Since a point of C_0 not in C is $\varepsilon/2$-close to a_1 or b_1, we get points on C that are $3\varepsilon/4$-close, while their distance on C is at least $c\|C\|/8$.

Finally, assume that no regular leaf goes from C to $C_1 \cup C_2$. Arguing as before, one shows first that a leaf going from C_i to itself cannot be essential and then that $|C_1|$ and $|C_2|$ are less than ε. Now simply apply Lemma III.4, with $P =$ the pair of pants.

Proof of (3) $\Rightarrow$ (2). Given $\varepsilon > 0$ and an integer $n \in \mathbf{N}$, we shall prove the following statement: *given four points on a closed curve C with $|C| \leqslant nc\varepsilon$, there exist points x, y on opposite arcs such that* $d_{\mathscr{F}}(x, y) \leqslant (1 - 2^{-n})\varepsilon$. This statement implies condition (2) by the usual compactness argument.

We fix ε and argue by induction on n (the case $n = 0$ being trivial). By Lemma III.4 (applied to an annulus), the result is true for $\|C\| < 2\varepsilon$. Assuming $\|C\| \geqslant 2\varepsilon$, consider $\mathscr{F}$-equivalent points $q, r \in C$ whose distance on C is at least $2c\varepsilon$.

If they belong to opposite arcs, there is nothing to prove. If not, one of the arcs qr or rq contains at most one of the four given points. Consider the curve C' we get from C after replacing this arc γ by an arc γ' between q and r, with $|\gamma'|_{\mathscr{F}} < \min(2^{-n}\varepsilon, c\varepsilon)$. If one of the four given points belongs to γ, replace it by any point of γ', so as to get four points on C'.

Note that $|C'| \leqslant |C| - |\gamma| + |\gamma'| \leqslant nc\varepsilon - 2c\varepsilon + c\varepsilon = (n-1)c\varepsilon$.

By the induction hypothesis, there exist points $x', y' \in C'$ on opposite arcs with $d_{\mathscr{F}}(x', y') \leqslant (1 - 2^{-n+1})\varepsilon$. Returning to C, we get points x, y on opposite arcs with $d_{\mathscr{F}}(x, y) \leqslant (1 - 2^{-n+1})\varepsilon + |\gamma'| \leqslant (1 - 2^{-n})\varepsilon$. ■

Let $N \subset \pi_1 M$ be a normal subgroup generated by elements of $\mathscr{F}$-length 0. To prove Theorem 1, we need to know that $\hat{T}(\mathscr{F})$ is an $\mathbf{R}$-tree under the following hypothesis (4 mod N): *given* $g \in \pi_1 M$ *and* $\varepsilon > 0$, *there exist* $g_1, g_2 \in \pi_1 M$ *such that* $g(g_1 g_2)^{-1} \in N$ *and*

$$\begin{cases} \|g_1\| + \|g_2\| < \|g\| + \varepsilon \\ \max(\|g_1\|, \|g_2\|) < (1-c)\|g\| + \varepsilon. \end{cases}$$

This condition is weaker than condition (4) since we only require $g = g_1 g_2 \bmod N$.

The proof is an extension of the one given above, replacing $\|g\|$ (and similarly $\|C\|$) by $\|g\|_N = \inf_{n \in N} \|gn\|$. Choose C and ε and let $g \in \pi_1 M$ be represented by C. Write $g = g_1 g_2 \bmod N$, with

$$\begin{cases} \|g_1\| + \|g_2\| < \|g\|_N + \varepsilon \\ \max(\|g_1\|, \|g_2\|) < (1-c)\|g\|_N + \varepsilon. \end{cases}$$

Then choose C_i representing $g_i \bmod N$, with $|C_i| < \|g_i\|_N + \delta$, and consider a compact surface of genus 0 bounded by $C \cup C_1 \cup C_2$ and possibly a family of other curves with very small total length. Arguing as above, one gets $\mathscr{F}$-equivalent points whose distance on C is at least $c'\|C\|_N$. The proof of (3) $\Rightarrow$ (2) then applies, using $\|C\|_N$ instead of $\|C\|$.

Proof of Theorem 1. We now show that $\widehat{T/H}$ is an $\mathbf{R}$-tree if ℓ satisfies (*).

As in the proof of Corollary III.6, we may assume that T satisfies the countability hypotheses of Theorem 4. Suppose that ℓ satisfies (*). Apply Theorem 4 to the action of H on T and let $N = \ker \rho \subset \pi_1 M$. Using assertion (4) of Theorem 4, one checks that $\mathscr{F}$ satisfies condition (4 mod N). It follows that $\hat{T}(\mathscr{F})$, and hence also $\widehat{T/H}$, is an $\mathbf{R}$-tree. ■

IV. Measured foliations on compact manifolds. In this section we first prove Theorem 6, and then Theorems 5 and 7.

Let $(M, \mathscr{F})$ be a Morse measured foliation, with M closed and $\dim M \geqslant 3$. Let $\mathscr{L}$, $\overline{\mathscr{L}}$, $M(\mathscr{L})$, $\mathscr{F}(\mathscr{L})$, f be as in the introduction. Let $d_{\mathscr{L}}$ be the pseudodistance defined by $\mathscr{F}(\mathscr{L})$ on $M(\mathscr{L})$ and let $\hat{T}(\mathscr{L})$ be the associated metric space. By Corollary III.5, we know that $\hat{T}(\mathscr{L})$ is an **R**-tree. Finally, define $J = \pi_1 M/\mathscr{L}$ and let $\psi: \pi_1 M \to J$ be the canonical epimorphism.

THEOREM IV.1. *Let $(M, \mathscr{F})$ be a Morse measured foliation, with M closed.*

(1) *Given $x, y \in M(\mathscr{L})$, there exists a path γ from x to y such that $|\gamma|_{\mathscr{F}(\mathscr{L})} = d_{\mathscr{L}}(x, y)$.*

(2) *Points of $\hat{T}(\mathscr{L})$ are in one-to-one correspondence with connected components of level sets $f^{-1}(c)$.*

(3) *Given $j \in J$, there exists a closed curve $C \subset M$ such that $\psi([C]) = j$ and $|C|_{\mathscr{F}} = \inf_{g \in \psi^{-1}(j)} \|g\|_{\mathscr{F}}$.*

(4) *If $g \in \pi_1 M$ and $\|g\|_{\mathscr{F}} = 0$, then $g \in \overline{\mathscr{L}}$.*

Theorem 6 is equivalent to assertion (2) of this theorem.

Proof of Theorem IV.1. The important thing to prove is (1): assertions (2), (3), and (4) then follow easily, using the fact that $\inf_{t \in \hat{T}(\mathscr{L})} d_{\mathscr{L}}(t, jt)$ is achieved.

For clarity, we introduce $M(\mathscr{L})^* = M(\mathscr{L}) - \operatorname{Sing} \mathscr{F}(\mathscr{L})$ and the nonsingular foliation $\mathscr{F}(\mathscr{L})^*$ induced on $M(\mathscr{L})^*$. Since $\pi_1(M(\mathscr{L})^*) \simeq \pi_1(M(\mathscr{L})) \simeq \mathscr{L}$ is generated by (free homotopy classes of) loops tangent to $\mathscr{F}(\mathscr{L})^*$, every leaf of $\mathscr{F}(\mathscr{L})^*$ separates $M(\mathscr{L})^*$ into two components. (In other words, the leaf space of $\mathscr{F}(\mathscr{L})^*$ is a *simply connected* non-Hausdorff one-manifold.)

LEMMA IV.2. *Let a, b, c be real numbers, with $a > 0$ and $b = \pm 1$. Let u and v be smooth maps from $[0, a]$ to $M(\mathscr{L})^*$ such that $(f \circ u)(\eta) = (f \circ v)(\eta) = b\eta + c$ for all η. Suppose $u(\eta)$ and $v(\eta)$ belong to the same leaf of $\mathscr{F}(\mathscr{L})^*$ if $\eta > 0$. Then $u(0)$ and $v(0)$ belong to the same connected component of $f^{-1}(c)$.*

Proof. Choose a J-invariant Riemannian metric on $M(\mathscr{L})$. On $M(\mathscr{L})^*$, let φ_t be the partially defined flow of the vector field $X = (\operatorname{grad} f)/\|\operatorname{grad} f\|^2$. (Note that $df(X) \equiv 1$, so that φ_t preserves $\mathscr{F}(\mathscr{L})^*$.) We may assume that $b = +1$, $c = 0$, and the images of u and v are contained in orbits of X.

Let σ be a smooth path joining $u(a)$ and $v(a)$ in their leaf. We use φ_{-t} to push σ onto neighboring leaves. If φ_{-a} is defined on σ, the lemma is clear. If not, let $t_0 \in (0, a]$ be the smallest t for which $\varphi_{-t}(\sigma)$ is not defined.

When t approaches t_0 from below, the path $\varphi_{-t}(\sigma)$ has a limiting position σ' in $M(\mathscr{L})$: σ' is a path tangent to $\mathscr{F}(\mathscr{L})$, passing through one or several singularities of index 1. (Singularities of index > 1 can be avoided by slightly moving σ, and singularities of index 0 cannot occur.)

If $t_0 = a$, the result is proved. If $t_0 < a$, let s (resp. s') be the first singularity one encounters when following σ' from $u(a - t_0)$ (resp. $v(a - t_0)$) and let $\mathscr{C}$ (resp. $\mathscr{C}'$) be the singular half-cone (see Section I) along which one arrives at s (resp. s').

By assumption, $\mathscr{C}$ and $\mathscr{C}'$ belong to the same leaf L of $\mathscr{F}(\mathscr{L})^*$, not just to the same component of $f^{-1}(a - t_0)$. This allows us to replace the part of σ' between s and s' by a path $c(\mathscr{C}, \mathscr{C}') \subset L$ connecting $\mathscr{C}$ to $\mathscr{C}'$, and keep pushing. If we can find some $\varepsilon > 0$, independent of $\mathscr{C}$ and $\mathscr{C}'$, such that $\varphi_{-\varepsilon}$ is defined on $c(\mathscr{C}, \mathscr{C}')$, then we can push until we reach $f^{-1}(0)$, thus proving the lemma.

Since we are working on the covering corresponding to $\mathscr{L}$, the projection of a leaf of $\mathscr{F}(\mathscr{L})^*$ onto its image in M is at most 2-to-1. By compactness, $\mathscr{F}$ has only finitely many singularities. It follows that the couples $(\mathscr{C}, \mathscr{C}')$ that may occur fall into finitely many orbits under the action of the group of covering transformations, whence the existence of ε. ∎

Remark. Similar arguments were used in [Le1, proof of lemma II.3] and [Le3, proof of lemma VIII.1]. Also note the resemblance between Lemma IV.2 and the notion of *segment closed* used in [Rim].

To prove assertion (1) of Theorem IV.1, join x to y by a piecewise smooth path τ, such that on each piece $f \circ \tau$ is either constant or strictly monotone. Let p be the number of pieces and let $k < p$ be the number of U-turns: k is the smallest integer such that the domain of τ can be subdivided into $k + 1$ intervals, with $f \circ \tau$ weakly monotone on each interval.

We argue by induction on (k, p) with the lexicographic order. If $k = 0$, then $f \circ \tau$ is weakly monotone and $d_{\mathscr{L}}(x, y) \leqslant |\tau|_{\mathscr{F}(\mathscr{L})} = |f(x) - f(y)| \leqslant d_{\mathscr{L}}(x, y)$, so that the result is true. If $k \geqslant 1$, choose τ between x and y with k minimal, and consider a U-turn. Fix $a_1 < a_2 \leqslant a_3 < a_4$ in the domain of τ, such that $f \circ \tau$ is strictly increasing (resp. decreasing) on $[a_1, a_2]$, constant on $[a_2, a_3]$, strictly decreasing (resp. increasing) on $[a_3, a_4]$, and $[a_1, a_2]$, $[a_3, a_4]$ are pieces of τ. Say for instance, $|f(a_1) - f(a_2)| \leqslant |f(a_3) - f(a_4)|$.

The set $Z \subset [a_1, a_2]$ consisting of all points z such that the leaf of $\mathscr{F}(\mathscr{L})^*$ containing $\tau(z)$ meets $\tau([a_3, a_4])$ is open and connected (because every leaf of $\mathscr{F}(\mathscr{L})^*$ separates $M(\mathscr{L})^*$). Let a_0 be its lower bound ($a_0 = a_2$ if $Z = \varnothing$). By Lemma IV.2, the point $\tau(a_0)$ can be joined to a point of $\tau([a_3, a_4])$ in a level set of f.

If $a_0 = a_1$, we can replace τ by a path with less pieces and no more than k U-turns. If $a_1 < a_0 < a_2$, minimality of k implies that the image of τ meets the leaf containing $\tau(a_0)$ exactly once, so that (the closure of) this leaf separates x from y. We then have $d_{\mathscr{L}}(x, y) = d_{\mathscr{L}}(x, \tau(a_0)) + d_{\mathscr{L}}(\tau(a_0), y)$, and the result follows since $\tau(a_0)$ can be joined to x and y by paths with strictly less than k U-turns. A similar argument works for $a_0 = a_2$. ∎

Remarks. This proof was inspired by an argument used jointly with P. Greenberg to prove the following. Let V be a Riemannian simply connected non-Hausdorff one-mainfold, and d the natural pseudodistance on V; if $d(v, v') = 0$, there exists a sequence $v_0 = v, v_1, \ldots, v_k = v'$ such that every neighborhood of v_i meets every neighborhood of v_{i+1}. (Note that Lemma IV.2 may be interpreted as saying when two points in the leaf space of $\mathscr{F}(\mathscr{L})^*$ do not have disjoint neighborhoods.) It seems quite likely that the metric space associated to (V, d) is always an $\mathbf{R}$-tree.

It is tempting to try to prove Theorem IV.1 in the universal covering $\tilde{M}$ rather than in $M(\mathscr{L})$, so that points of $T(\tilde{\mathscr{F}})$ would correspond to components of level sets $\tilde{f}^{-1}(c)$. Unfortunately, the finiteness argument that concludes the proof of Lemma IV.2 does not apply. One situation where it does work is when each critical point of index 1 or $n - 1$ of $\tilde{f}$ has a connecting loop (in the sense of [Le1]); this happens for instance if $\pi_1 M$ is virtually polycyclic, by [Le1, cor. 5.4].

Proof of Theorem 5. We fix three positive, rationally independent, numbers α, β, γ, and we construct a free nonsimplicial action of F_3 on an **R**-tree such that α, β, γ are the translation lengths of the generators.

Let $C = \mathbf{R}/\mathbf{Z}$ be a circle of length 1. Consider two open intervals $I_1 = (a_1, a_1 + \ell_1)$ and $I_2 = (a_2, a_2 + \ell_2)$, with $0 < \ell_1, \ell_2 < 1$. Let α_1, α_2 be two real numbers. Let γ_i be the restriction to I_i of the rotation $x \mapsto x + \alpha_i \bmod 1$, and $J_i = \gamma_i(I_i)$.

This data yields a Morse measured foliation $\mathscr{F}$ by the same construction as in Section II. Instead of starting from $\mathbf{R}^{n-1} \times \mathbf{R}$, we use $N \times C$, with N a simply connected closed manifold (e.g. S^2), so that we get a closed manifold M with $\pi_1 M \simeq F_3$. The foliation $\mathscr{F}$ is transversely orientable, and it has four singularities (two of index 1, and two of index $n - 1$). Recall from the introduction that we want all leaves dense and $\overline{\mathscr{L}} = \{1\}$.

Leaves of $\mathscr{F}$ are in one-to-one correspondence with the *orbits of the pseudogroup* Γ generated by γ_1 and γ_2, i.e. the classes of the equivalence relation generated by "$x \sim y$ if $x \in I_i$ and $y = x + \alpha_i \bmod 1$ ($i = 1$ or 2)". In particular, all leaves of $\mathscr{F}$ are dense if and only if Γ is *minimal*, i.e. all its orbits are dense.

It is easy to construct minimal pseudogroups, but in general they will be equivalent to a group in the sense of [Le3]. The group $\pi_1 M/\mathscr{L}$, isomorphic to the group $\pi_1(C/\Gamma)$ used in [Le3], is then free abelian. Minimal pseudogroups with $\pi_1(C/\Gamma) \simeq F_3$ are found by requiring $\ell_1 + \ell_2 = 1$.

THEOREM ([Le3, théorème 1]). *Fix α_1 and α_2, with $1, \alpha_1, \alpha_2$ rationally independent. Given a_1 and a_2, there exist uncountably many $\ell \in (0, 1)$ such that the pseudogroup Γ determined by $\alpha_1, \alpha_2, a_1, a_2, \ell_1 = \ell, \ell_2 = 1 - \ell$ is minimal. These minimal pseudogroups are not equivalent to a group* (*so that* $\pi_1(C/\Gamma) \simeq F_3$).

The foliation corresponding to such a Γ satisfies $\mathscr{L} = \{1\}$. To guarantee that $\overline{\mathscr{L}} = \{1\}$, it is enough to ensure that there is no connection between singularities of $\mathscr{F}$ (cf. [Le2, figure 1]). In terms of Γ, this means that no orbit meets two of the pairs $\{a_1, a_1 + \alpha_1\}, \{a_2, a_2 + \alpha_2\}, \{a_1 + \ell, a_1 + \ell + \alpha_1\}, \{a_2 + 1 - \ell, a_2 + 1 - \ell + \alpha_2\}$. (Two points in the same pair always have distinct orbits, but we do not need this.) The above theorem allows us to achieve this, by choosing a_1, a_2, ℓ so that $1, \alpha_1, \alpha_2, a_1 - a_2, \ell$ are rationally independent.

Finally, we need to control translation lengths. Let α, β, γ be as in the statement of Theorem 5. We may assume $1 = \alpha > \beta > \gamma$. Perform the preceding construction with $\alpha_1 = \beta$ and $\alpha_2 = \gamma$. Note that the natural generators of $\pi_1 M$ can be represented (as free homotopy classes) by closed curves transverse to $\mathscr{F}$ with respective $\mathscr{F}$-lengths $1, \alpha_1, \alpha_2$. Since $\mathscr{F}$ is transversely orientable, a transverse curve minimizes $\mathscr{F}$-length in its free homotopy class. It follows that, for the free action of $\pi_1 M$ on $T(\tilde{\mathscr{F}})$ given by the corollary of Theorem 6, the generators have translation lengths $1, \alpha_1, \alpha_2$. ■

Proof of Theorem 7. We prove that, if a Morse measured foliation $(M, \mathscr{F})$ on a closed manifold is transversely orientable, then $\pi_1 M/\overline{\mathscr{L}}$ acts freely on the **R**-tree $\hat{T}(\overline{\mathscr{L}})$.

Let $M(\bar{\mathscr{L}})$ be the covering of M corresponding to $\bar{\mathscr{L}}$. Since $\mathscr{F}$ is assumed to be transversely orientable, the induced foliation $\mathscr{F}(\bar{\mathscr{L}})$ can be defined by a Morse function $\bar{f}: M(\bar{\mathscr{L}}) \to \mathbf{R}$. Let $\bar{d}$ be the associated pseudodistance on $M(\bar{\mathscr{L}})$. It suffices to prove on $M(\bar{\mathscr{L}})$ the analogue of assertion (1) of Theorem IV.1: $\bar{d}(x, y)$ is always realized by a path. The arguments are quite similar to those used before.

Define $Z \subset [a_1, a_2]$ as in the proof of Theorem IV.1. It may have more than one component, because a leaf of $\mathscr{F}(\bar{\mathscr{L}})^*$ whose closure contains a singularity of index 1 or $n-1$ may fail to separate $M(\bar{\mathscr{L}})^*$.

LEMMA IV.3. *Let L be a leaf of $\mathscr{F}(\bar{\mathscr{L}})^*$. Let W^+ (resp. W^-) be the set of all points of $M(\bar{\mathscr{L}})$ that can be joined to L by a path tangent to $\mathscr{F}(\bar{\mathscr{L}})$ avoiding all critical points of index* 1 *(resp. $n-1$). Then W^+ (resp. W^-) separates $M(\bar{\mathscr{L}})$ into at least two components.*

Proof. We give the proof for $W = W^+$. We check that W has intersection number 0 with any loop γ tangent to $\mathscr{F}(\bar{\mathscr{L}})$. The only case to consider is when γ meets W. To compute the intersection number, we must push γ into $M(\bar{\mathscr{L}})^*$ and make it transverse to W. This can be done in such a way that $\bar{f}|\gamma$ is greater than $\bar{f}(L)$, except near critical points of index $n-1$. The intersection of γ and W then consists of pairs of points located near singularities of index $n-1$. The two points of each pair give contributions of opposite sign to the intersection number, so that $W \cdot \gamma = 0$. ■

It follows from Lemma IV.3 that Z consists of adjacent intervals. Arguing as in the proof of Lemma IV.2, one sees that Z has only finitely many components. Furthermore, $\tau(a_0)$ can be joined to a point of $\tau([a_3, a_4])$ in a level set of $\bar{f}$. If $a_0 > a_1$, then one of the two sets W^+ or W^- associated to the leaf of $\tau(a_0)$ separates x from y, so that we can use the same induction argument as in the proof of Theorem IV.1. ■

Remark IV.4. Any finitely generated pseudogroup of rotations Γ on the circle $C = \mathbf{R}/\mathbf{Z}$ gives rise (as in the proof of Theorem 5) to a transversely orientable Morse measured foliation on a closed manifold M, and hence to a free action of a group $G(\Gamma)$ $(=\pi_1 M/\mathscr{L})$ on an $\mathbf{R}$-tree. Going back to the circle, Theorem 7 leads to the following presentation of $G(\Gamma)$ (cf. the remark ending part III of [Le3]). Let Γ be generated by partially defined rotations $\gamma_i: I_i \to J_i$ $(1 \leqslant i \leqslant k)$, as in the proof of Theorem 5. Let $\tilde{\gamma}_i: \tilde{I}_i \to \tilde{J}_i$ be a lift of γ_i to the universal covering $\tilde{C} \simeq \mathbf{R}$, and $\bar{\gamma}_i: \bar{I}_i \to \bar{J}_i$ the extension of γ_i to the closure of I_i. Also let $\bar{\gamma}_0: \mathbf{R} \to \mathbf{R}$ be the unit translation. Then $G(\Gamma)$ is generated by elements t_i $(0 \leqslant i \leqslant k)$, relators being the words $t_{i_1}^{\varepsilon_1} \cdots t_{i_p}^{\varepsilon_p}$ such that there exists $x \in \mathbf{R}$ with $\bar{\gamma}_{i_1}^{\varepsilon_1} \circ \cdots \circ \bar{\gamma}_{i_p}^{\varepsilon_p}(x) = x$.

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