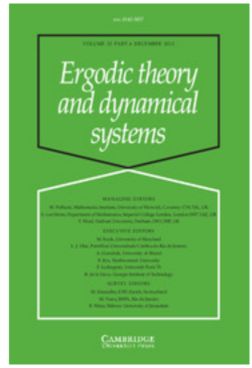

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Pseudogroups of isometries of $\mathbb{R}$: reconstruction of free actions on $\mathbb{R}$-trees

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Abstract. Rips' theorem about free actions on $\mathbb{R}$-trees relies on a careful analysis of finite systems of partial isometries of $\mathbb{R}$. We associate a free action on an $\mathbb{R}$-tree to any finite system of isometries without reflection. Any free action may be approximated (strongly in the sense of Gillet–Shalen) by actions arising in this way. Proofs involve the use, in an essential way, of separation properties of systems of isometries. We also interpret these finite systems of isometries as generating sets of pseudogroups of partial isometries between closed intervals of $\mathbb{R}$.

0. *Introduction*

The theory of one-dimensional dynamical systems involves the study of the iterations of selfmap(s) of a subset of the line. We are interested in the isometric case with some finiteness assumptions. More precisely we consider *systems of isometries* $X = (D, \{\phi_j\})$, where D is the disjoint union of a finite number of compact intervals, and $\{\phi_j\}$ is a finite set of partial isometries between closed intervals in D. Using closed intervals, instead of the more usual open ones, causes some difficulties (for instance for separation properties), but it is more convenient for certain applications.

These systems arise for instance from transversely measured codimension 1 foliations on compact manifolds, by taking transversals and first return maps (see for instance [**Hae1, Hae2**]). As shown by Rips, who named them Makanin Combinatorial Objects, they also arise in the study of actions of finitely generated groups G on $\mathbb{R}$-trees T (see [**GLP1**] and §4). Given a finite subtree K of T, one gets a system of isometries $X(K)$ by taking the restrictions to K of generators of G, and splitting K open at branch points to turn it into a union of intervals. Recall that an $\mathbb{R}$-tree is an arcwise connected metric space in which every arc is isometric to an interval of $\mathbb{R}$. See for instance [**Sha1, Sha2, Mor1**] for historical remarks, references and motivations on $\mathbb{R}$-trees.

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There is a group $G(X)$ naturally associated with X. When D is connected, it has the following presentation (see [**Lev1**]): the generators are in one-to-one correspondence with the ϕ_j's, and the relations correspond to words in the ϕ_j's and their inverses having a fixed point in D.

In [**GLP1**] we gave a proof of Rips's result that $G(X)$ is a free product of free abelian groups and surface groups. This is the key step in proving that a finitely generated group G acting freely on an $\mathbb{R}$-tree T is such a free product; if the finite subtree $K \subset T$ is big enough, then $G(X(K))$ is isomorphic to G.

In this article we show how to associate a free action of $G(X)$ on an $\mathbb{R}$-tree $T(X)$ with any system of isometries *without reflection* (see §3). This applies in particular to systems $X(K)$ obtained as above from a free action on an $\mathbb{R}$-tree T. In this case we show that the trees $T(X(K))$ are strong approximations of the original tree T when K is big (see §4).

The tree $T(X)$ may be viewed geometrically as follows. There is a compact foliated 2-complex $\Sigma(X)$ (see §1) canonically associated with X; it is obtained by attaching strips to D, one for each ϕ_j. The group $G(X)$ is obtained from $\pi_1\Sigma(X)$ by killing all loops contained in leaves. Let $\overline{\Sigma}(X)$ be the covering of $\Sigma(X)$ with transformation group $G(X)$.

The group $G(X)$ acts freely on the space of leaves of the foliation induced on $\overline{\Sigma}(X)$. This space has a natural pseudo-metric d (coming from length on D) and the associated metric space $T(X)$ is an $\mathbb{R}$-tree (see [**Lev3**, Corollary III.5]). But d may be quite far from being a metric (it may even be identically 0), so that the natural isometric action of $G(X)$ on $T(X)$ is not always free (from the point of view of [**Lev3**], this means that $\overline{\Sigma}(X)$ is not the right covering space to consider!).

We thus have to impose some condition on X to rule out such pathologies. Say that X is *without reflection* if no partial isometry obtained by composing the generators φ_j and their inverses carries an interval $[x-u, x]$ $(u > 0)$ onto $[x, x+u]$ in an orientation-reversing way.

THEOREM 3.2. *If X is a system without reflection, then d is a metric and the natural action of $G(X)$ on the $\mathbb{R}$-tree $T(X)$ is free.*

Now suppose G is a finitely generated group acting freely and minimally on an $\mathbb{R}$-tree T. Let K_n be an increasing sequence of finite subtrees, with $T = \bigcup K_n$. Then $G(X(K_n))$ is isomorphic to G for n big enough (see [**GLP1**]), and it is easy to see that the systems $X(K_n)$ have no reflection. We thus get a sequence of $\mathbb{R}$-trees $T(X(K_n))$, each equipped with a free action of G.

THEOREM 4.3. *The sequence $T(X(K_n))$ converges towards T strongly (in the sense of* [**GS**]*), hence also in the equivariant Gromov topology (in the sense of* [**Pau2**]*).*

The key point in Theorem 3.2 is that d is a metric (i.e. the space of leaves on $\widetilde{\Sigma}(X)$ is Hausdorff). This is proved using a fact about systems of isometries established in §2, generalizing Lemma VIII.1 of [**Lev2**].

Given X, say that two points x, y in D are in the same X-orbit if there is a word in the ϕ_j's and their inverses defined at x with image y (we call such a word an X-word).

THEOREM 2.3. *Let X be a system of isometries. Suppose p, q are isometric embeddings $[0, \eta] \to \mathbb{R}$ with $\eta > 0$ such that $q_t = q(t)$ and $p_t = p(t)$ are in the same X-orbit for all but countably many t's. Then q_t and p_t are in fact in the same X-orbit for every t. Furthermore there exist finitely many X-words $\omega_1, \dots, \omega_n$ such that for every $t \in [0, \eta]$ there is an $i \in \{1, \dots, n\}$ with $\omega_i(p_t) = q_t$.*

This implies that a system of isometries is *segment closed* (a property introduced by Rimlinger [**Rim1**]), see §2.

Theorem 2.3 has several other consequences. In §5, we interpret the systems X as generating systems of finitely generated *closed pseudogroups* $\mathcal{P}$ on D. That is, $\mathcal{P}$ is the smallest set of partial isometries between closed subintervals of D that contains the ϕ_j's and is stable under the following operations: composition, passage to the inverse, restriction and finite glueing. There is a notion of equivalence of closed pseudogroups (for instance different choices of a complete transversal for a measured foliation yield equivalent pseudogroups).

We prove (Proposition 5.9) that if two systems X, X' generate equivalent closed pseudogroups, then the associated groups $G(X)$, $G(X')$ are isomorphic (see [**Rim3**] for a partial result) and the trees $T(X)$, $T(X')$ are isometric.

Every orbit of a system X is the set of vertices of a graph, called the *Cayley graph of the orbit*, where there is an edge labelled j between x, y whenever $y = \phi_j(x)$. We view this graph as a metric space, by giving length 1 to every edge.

If systems X and X' on a given multi-interval D have the same orbits, then the orbits of a given $x \in D$ for X and X' are *quasi-isometric* (see Proposition 5.6). This follows from Theorem 2.3 since X admits a finite refinement X_0 such that every generator of X_0 may be expressed as an X'-word. It is interesting to study properties of the orbits (such as growth or number of ends), see [**Gab**] for results and examples.

This article is a sequel to [**GLP1**], but it may be read independently.

1. *Notation and definitions*

For the sake of completeness, we recall here notations and definitions from [**GLP1**].

A *multi-interval* D is a union of finitely many disjoint compact intervals of $\mathbb{R}$. Components of D may be *degenerate intervals*, i.e. consist of only one point.

Definition 1.1. A system of isometries is a pair $X = (D, \{\varphi_j\}_{j=1,\dots,k})$, where D is a multi-interval and each $\varphi_j : A_j \to B_j$ (called a generator) is an isometry between closed (possibly degenerate) subintervals of D.

The intervals A_j, B_j are called *bases*. A generator $\varphi_j : A_j \to B_j$ is *a singleton* if A_j is degenerate.

An *X-word* is a word in the generators $\varphi_j^{\pm 1}$. It is a partial isometry of D, whose domain (defined in the obvious maximal way) is a closed interval (possibly degenerate or empty). If φ_j is not a singleton, define $\mathring{\varphi}_j : \mathring{A}_j \to \mathring{B}_j$ as the restriction of φ_j to the interior of A_j. An $\mathring{X}$-word is a word in the generators $\mathring{\varphi}_j^{\pm 1}$. Its domain is a (possibly empty) open interval.

Two points x, y in D *belong to the same X-orbit* (resp. *$\mathring{X}$-orbit*) if there exists an X-word (resp. $\mathring{X}$-word) sending one to the other. Note that the orbits are countable and that an orbit of $\mathring{X}$ is contained in an orbit of X, with equality except perhaps for a finite number of them. Every orbit of X may be viewed as a 'Cayley graph': the vertices are the elements of the orbit, and there is an edge labelled j between x and y whenever $y = \varphi_j(x)$.

We can associate a sign $\pm$ with every partial isometry of $\mathbb{R}$ whose domain has non-empty interior, simply by taking its derivative.

If X is a system of isometries on a multi-interval D, we define (see [**GLP1**, §1]) a *foliated* 2*-complex* $(\Sigma(X), \mathcal{F})$ (or simply Σ) associated with X. Start with the disjoint union of D (foliated by points) and strips $A_j \times [0, 1]$ (foliated by $\{*\} \times [0, 1]$). We get Σ by glueing the strips $A_j \times [0, 1]$ to D, identifying each $(t, 0) \in A_j \times \{0\}$ with $t \in A_j \subset D$ and each $(t, 1) \in A_j \times \{1\}$ with $\varphi_j(t) \in B_j \subset D$. We will identify D with its image in Σ.

The foliation $\mathcal{F}$ is the decomposition of Σ into leaves. A leaf is an equivalence class for the equivalence relation $\sim$ generated by $x \sim y$ if there is a $j = 1, \dots, k$ with x, y corresponding to two points in the same leaf $\{*\} \times [0, 1]$ of $A_j \times [0, 1]$. Each leaf L is a simplicial 1-complex (whose embedding in Σ may fail to be proper): the vertices are the intersections of L with D, and the edges correspond to the leaves $\{*\} \times [0, 1]$ of the strips $A_j \times [0, 1]$. Two points of D are in the same leaf of $\mathcal{F}$ if and only if they are in the same X-orbit. For instance, if a point $x \in D$ does not belong to any base, then its leaf is just $\{x\}$.

This suspension process is well-known for interval exchanges. See Morgan's notes [**Mor2**] for the first appearance under the above generality, and [**AL**] and [**Lev3**] for suspensions as measured foliations with Morse singularities on manifolds.

In what follows, we assume that Σ is connected. The 2-complex Σ has the homotopy type of a finite graph, so that its fundamental group $\pi_1(\Sigma)$ is a finitely generated free group. We will denote by $\overline{\mathcal{L}}$ the normal subgroup of $\pi_1(\Sigma)$ normally generated by the free homotopy classes of loops contained in leaves of $\mathcal{F}$.

Definition 1.2. If X is a system of isometries, we define $G(X) = \pi_1(\Sigma)/\overline{\mathcal{L}}$.

2. *The segment closed property*

When one wants to study separation properties of the quotient of a metric space by an isometric equivalence relation, the following notion, introduced by Rimlinger [**Rim1**], [**Rim2**], may be considered.

Definition 2.1. Let $\mathcal{R}$ be an equivalence relation on a metric space M. Then $\mathcal{R}$ is segment closed if, given an isometry φ between geodesic segments $[a, c]$ and $[b, d]$ in M sending a to b, with $\varphi(t)$ equivalent to t for every $t \in (a, c]$, then a and b are equivalent.

PROPOSITION 2.2. *If X is a system of isometries, the equivalence relation on D defined by 'being in the same X-orbit' is segment closed.*

More precisely:

THEOREM 2.3. *Let X be a system of isometries. Suppose p, q are isometric embeddings $[0, \eta] \to \mathbb{R}$ with $\eta > 0$ such that $q_t = q(t)$ and $p_t = p(t)$ are in the same X-orbit for all* *but countably many t's. Then q_t and p_t are in the same X-orbit for every t. Furthermore, there exist finitely many X-words $\omega_1, \ldots, \omega_n$ such that for every $t \in [0, \eta]$ there is an $i \in \{1, \ldots, n\}$ with $\omega_i(p_t) = q_t$.*

First a few definitions, for X a system of isometries. Let $X = \{\varphi_j : [a_j, c_j] \to [b_j, d_j]\}_{j=1,\ldots,k}$. Implicit in this notation is the fact that $\varphi_j(a_j) = b_j$, so that we allow $a_j > c_j$ or $b_j > d_j$. Define Ω to be the set of points a_j, c_j, b_j, d_j. Let Γ_X be the (non-oriented) graph having as vertices the $\mathring{X}$-orbits meeting Ω, and having two edges for each $j = 1, \ldots, k$, one edge between the $\mathring{X}$-orbit of a_j and the $\mathring{X}$-orbit of b_j, and another edge between the $\mathring{X}$-orbit of c_j and the $\mathring{X}$-orbit of d_j. Define D_X to be the number of vertices of Γ_X minus the number of connected components of Γ_X.

Say that $x \in \mathbb{R}$ is *bad* for X if there is no $\mathring{X}$-word w of derivative -1 with $w(x) = x$, but there is an $\mathring{X}$-word w of the form $(x - u, x) \to (x + u, x)$ with $u > 0$. Say that X is *good* if it has no bad point. Note that if X has a bad point, then some point of Ω is bad. Indeed, every point in the $\mathring{X}$-orbit of a bad point is also bad. But endpoints of domains and ranges of $\mathring{X}$-words are in $\mathring{X}$-orbits meeting Ω.

LEMMA 2.4. *For every system of isometries X, there exists a good system of isometries X' such that X and X' (resp. $\mathring{X}$ and $\mathring{X}'$) have the same orbits, and $D_X = D_{X'}$.*

Proof. If $x \in \Omega$ is a bad point for X, with an $\mathring{X}$-word w of the form $(x - u, x) \to (x + u, x)$, we add a generator $[x - \delta, x + \delta] \to [x + \delta, x - \delta]$ of derivative -1, with $\delta < u$. The orbits of X and X' (resp. $\mathring{X}$ and $\mathring{X}'$) are the same, where X' is the new system of isometries. By choosing δ outside a countable set, we may assume that $x \pm \delta$ does not belong to the X-orbit of a point of Ω. In particular, there is no $\mathring{X}$-word w of the form $(x + \delta - u, x + \delta) \to (x + \delta + u, x + \delta)$ with $u > 0$. Hence, the points $x \pm \delta$ are not bad for X'.

The graph $\Gamma_{X'}$ is obtained from Γ_X by adding a component with one vertex and two edges, so that D_X has not changed. By iterating, one gets the result. □

The following proposition is a statement analogous to Theorem 2.3 in the open case (compare [**Lev2**, Lemma VIII.1]).

PROPOSITION 2.5. *Let X be a good system of isometries. There exists $\delta > 0$ such that if p, q are isometric embeddings $[0, \eta'] \to \mathbb{R}$ with $0 < \eta' < \delta$ such that $q_t = q(t)$ is in the $\mathring{X}$-orbit of $p_t = p(t)$ for every $t \in (0, \eta')$, then there exists an $\mathring{X}$-word sending $(p_0, p_{\eta'})$ to $(q_0, q_{\eta'})$. In particular, p_0 and q_0 are in the same X-orbit.*

Proof. For every u, v in Ω and $\mu = \pm 1$ such that there exists an $\mathring{X}$-word sending u to v with derivative μ, we fix such a word $w_{u,v,\mu}$. Since Ω is finite, we may find $\delta > 0$ such that each $w_{u,v,\mu}$ is defined on $(u - \delta, u + \delta)$.

Define λ to be $+1$ if the orientations of p and q coincide, and -1 otherwise. For all but countably many $t \in (0, \eta')$, there exists an $\mathring{X}$-word w with derivative λ such that $w(p_t) = q_t$: there are countably many $\mathring{X}$-words, and such a word can send at most one p_t to q_t with the wrong orientation.

Consider $\mathring{X}$-words $w = \varphi_{i_1} \cdots \varphi_{i_n}$ with derivative λ, such that the domain of w meets $(p_0, p_{\eta'})$ and $w(p_t) = q_t$ for some $t \in (0, \eta')$. Define by induction $x_t^0 = q_t, \varphi_{i_j}(x_t^j) =$

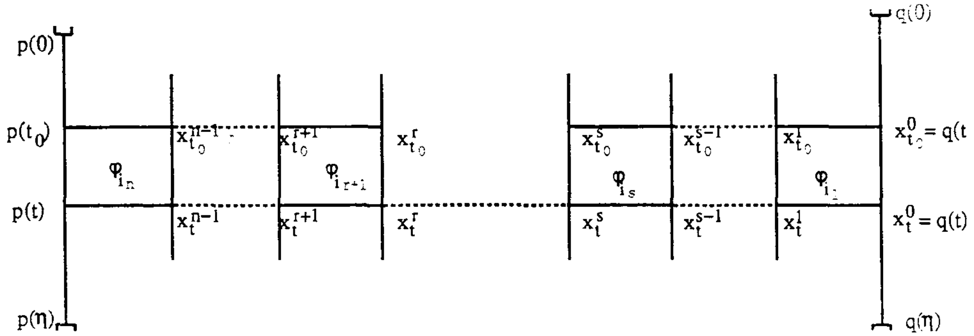

FIGURE 1. Pushing part of an orbit into nearby ones

x_t^{j-1}, so that $p_t = x_t^n$. For every $0 \leq t' \leq \eta'$, let $x_{t'}^i$ be the point of $\mathbb{R}$ at distance $|t-t'|$ from x_t^i and on the correct side (for instance if $q_0 < q_t$ and φ_{i_1} preserves the orientation then $x_0^1 < x_t^1 < x_{\eta'}^1$).

Let $c(w)$ be the number of $j \in \{1, \ldots, n\}$ such that φ_{i_j} is not defined on the whole open interval between x_0^i and $x_{\eta'}^i$. Choose w with $c(w)$ minimal. If $c(w) = 0$, then we are done. Otherwise, since the set of t such that $w(p_t) = q_t$ is an open interval, there is a biggest $t_0 \in [0, t)$ such that w is not defined at p_{t_0} and a smallest $t_0' \in (t, \eta']$ such that w is not defined at $p_{t_0'}$.

If $t_0 = 0$ and $t_0' = \eta'$, then we are done. So suppose for instance that $t_0 > 0$. Let $s \geq 0$ be the smallest j such that $\varphi_{i_{j+1}}^{-1}(x_{t_0}^j)$ is not defined. Let $r \leq n$ be the biggest j such that $\varphi_{i_j}(x_{t_0}^j)$ is not defined. Observe that both $v = x_{t_0}^s$ and $u = x_{t_0}^r$ are in Ω. Moreover, we have $0 \leq s < r \leq n$ since otherwise w would be defined at p_{t_0}.

Let w_* be the subword of w taking x_t^r to x_t^s for $t > t_0$ close enough to t_0, and let μ be its derivative. Since q_{t_0} is in the same $\mathring{X}$-orbit as p_{t_0}, there is an $\mathring{X}$-word w' sending u to v. We claim that w' may be chosen to have derivative μ.

If w' has the wrong derivative $-\mu$, then by composing it with the inverse of w_*, we get a map of the form $(u - \nu, u) \to (u + \nu, u)$ with $\nu > 0$. Since X is good, we can correct w' by using an $\mathring{X}$-word with derivative -1 fixing u.

Replacing the subword w_* in w by $w_{u,v,\mu}$, we now get a contradiction to the minimality of $c(w)$, since $\eta' < \delta$. □

Proof of Theorem 2.3. We prove that there exist finitely many X-words $\omega_1, \ldots, \omega_n$ such that for every $t \in [0, \eta]$ there is an $i \in \{1, \ldots, n\}$ with $\omega_i(p_t) = q_t$. In particular, q_t and p_t are in the same X-orbit for every t.

Define λ to be $+1$ if the orientations of p and q coincide, and -1 otherwise. Let $F = F(X) \subset [0, \eta]$ be the set of t such that there is no $\mathring{X}$-word with derivative λ taking p_t to q_t.

Clearly F is closed, since domains of $\mathring{X}$-words are open and orientations agree. Moreover, F is countable: q_t is in the same $\mathring{X}$-orbit as p_t for all but countably many t's (the orbits for X and $\mathring{X}$ differ only for countably many points) and the derivative is right for all but countably many t's.

If X is good and F is empty, we are done by Proposition 2.5. If not, we now construct a good system Y with the same orbits as X and $F(Y)$ empty.

We make X good by Lemma 2.4 and assume F is non-empty. Let $t_0 \in (0, \eta)$ be an isolated point of F (recall that a non-empty closed countable subset of $\mathbb{R}$ has an isolated point).

If there is an $\mathring{X}$-word w sending p_{t_0} to q_{t_0}, then w has derivative $-\lambda$ by definition of F. Applying Proposition 2.5 on the right or on the left of p_{t_0} and composing by w^{-1}, we see that there is an $\mathring{X}$-word of the form $(p_{t_0} - u, p_{t_0}) \to (p_{t_0} + u, p_{t_0})$ with $u > 0$. Since X is good, there is a reflection around p_{t_0}. Composing it with w contradicts $t_0 \in F$.

The $\mathring{X}$-orbits of p_{t_0} and q_{t_0} are thus distinct. Construct a new system of isometries X' by adding to X a new generator $[p_{t_0-u}, p_{t_0+u}] \to [q_{t_0-u}, q_{t_0+u}]$ with $u > 0$ chosen so that the $\mathring{X}$-orbits of p_{t_0-u} and q_{t_0+u} do not meet Ω. By Proposition 2.5, orbits for X and X' are the same if u is small enough.

We claim that $D_{X'} = D_X - 1$. By Proposition 2.5, the $\mathring{X}$-orbits of p_{t_0} and q_{t_0} both meet Ω. So one goes from Γ_X to $\Gamma_{X'}$ by identifying two vertices belonging to the same component, and adding two components with one vertex and one edge (if the $\mathring{X}$-orbits of p_{t_0-u} and q_{t_0+u} are the same) or one component with one vertex and two edges (otherwise), so that D always decreases.

Making X' good by Lemma 2.4 and iterating finitely many times (D is non-negative), we obtain a good system of isometries Y having the same orbits as X, with $F(Y)$ empty.

To conclude the proof, we now observe that the result is true for Y by Proposition 2.5, and that all new generators φ of Y (introduced either in the proof of Lemma 2.4 or in the proof above) have the following property: their domain is a finite union of closed intervals I_i such that on I_i the map φ agrees with the restriction of some X-word. □

3. *The $\mathbb{R}$-tree associated to a system of isometries*

We define an action of a group on a metric space to be a left isometric action.

Let X be a system of isometries on a multi-interval D. We assume that the associated foliated 2-complex $(\Sigma(X), \mathcal{F})$ is connected, and we consider the group $G(X) = \pi_1\Sigma(X)/\overline{\mathcal{L}}$.

We define a metric space $T(X)$ as follows (see for instance [**Lev3**]). Let $\pi : \overline{\Sigma}(X) \to \Sigma(X)$ be the covering defined by $\overline{\mathcal{L}}$, and $\overline{\mathcal{F}}$ be the measured foliation lifting $\mathcal{F}$. A path γ in $\overline{\Sigma}(X)$ has a *length* $\|\gamma\|$, defined as the total mass of the measure induced on γ by the transverse measure of $\overline{\mathcal{F}}$. Given x, y in $\overline{\Sigma}(X)$, let $d_{\overline{\mathcal{F}}}(x, y)$ be the infimum of the lengths of all paths from x to y. This defines a pseudo-distance on $\overline{\Sigma}(X)$ (two different points x, y may have $d_{\overline{\mathcal{F}}}(x, y) = 0$, for instance if they are in the same leaf).

The metric space $T(X)$ is obtained by identifying two points in $\overline{\Sigma}(X)$ at pseudo-distance 0 from each other. Since $\overline{\mathcal{F}}$ is invariant by $G(X)$, there is a natural (isometric) action of $G(X)$ on $T(X)$. The natural projection $\theta : \overline{\Sigma}(X) \to T(X)$ is equivariant and continuous.

Using ideas from [**GS** Theorem 5.20], [**Pau3** Proposition 4.6], the second author has proved the following general result.

PROPOSITION 3.1. ([**Lev3** Corollary III.5].) *If X is a system of isometries, then $T(X)$ is* *an $\mathbb{R}$-tree.*

We will reprove it when X is without reflection. A *reflection* in X is a negative X-word having a fixed point, called its *center* (recall that a negative word is a word with non-degenerate domain and derivative -1). According to Theorem 2.3, a system of isometries is without reflection if and only if there is no x such that $x+t$ is in the orbit of $x-t$ for $t>0$ small. Note that a system of isometries without reflection may still have negative words.

THEOREM 3.2. *If X is a system of isometries without reflection, then $G(X)$ acts freely on the $\mathbb{R}$-tree $T(X)$. Moreover, $T(X)$ is the space of leaves of $\overline{\mathcal{F}}$, that is $d_{\overline{\mathcal{F}}}(x, y) = 0$ if and only if x, y are in the same leaf.*

Eliminating reflections (called 'folds') is the main technical problem in [**MS1**] and the ideas of the above theorem may be found therein and in [**MO**].

Note that the same result is proved in [**Lev3**, Theorem 7], under the stronger hypothesis that $\overline{\mathcal{F}}$ is transversely orientable.

Remark. Arnoux and Yoccoz [**AY**] have constructed examples of singular measured foliations on the projective plane such that every leaf is dense and simply connected. Taking transversals, one gets a system of isometries X (having infinitely many centers of reflections) such that the associated group is $G(X) = \mathbb{Z}/2\mathbb{Z}$ and $T(X)$ is a point (see [**Lev3**]). This shows that the absence of reflection is not a superfluous condition (though it may be relaxed to almost without reflection, see Theorem 5.13).

3.1. *General position of surfaces in 2-complexes.* Before giving the proof of Theorem 3.2, we need to define what it means for a map f of a surface with boundary S into a foliated (finite) cellular 2-complex Σ to be in general position.

First, a map γ from a 1-complex K into Σ is in a general position if it is PL (see for instance [**RS**]) and every 1-simplex may be finitely subdivided so that every subinterval maps injectively and is either transverse or contained in a leaf. Looking at the local models, we see that every path in Σ may be homotoped (relatively to endpoints) to a nearby path in general position (without increasing the total mass if the foliation is equipped with a transverse measure).

Second, a map from a compact surface into a foliated 2-complex is in a general position if, as a map from the surface to the 2-complex, it is PL (see for instance [**RS**]) and the restriction to the 1-skeleton is in general position. Any continuous map from a compact surface into a foliated 2-complex whose restriction to the boundary is in general position may be homotoped relatively to its boundary to a map in general position. The preimage of the foliation defines (up to subdivision) a foliated 2-complex structure on the surface.

Proof of Theorem 3.2. We consider the pseudometric $d_{\overline{\mathcal{F}}}$ on $\overline{\Sigma}(X)$. We are going to prove two assertions:

(1) given $x, y \in \overline{\Sigma}(X)$, there exists a path γ from x to y such that $\|\gamma\| = d_{\overline{\mathcal{F}}}(x, y)$.
(2) the metric space $T(X)$ associated to $d_{\overline{\mathcal{F}}}$ is an $\mathbb{R}$-tree.

The fact that the action of $G(X)$ on $T(X)$ is free then follows. Indeed, if $g \in G(X)$ has a fixed point $\theta(x)$ in $T(X)$, then $d_{\overline{\mathcal{F}}}(x, gx) = 0$ so that x and gx are in the same leaf by (1). Since the covering is defined by $\overline{\mathcal{L}}$, this implies that g is the identity.

Say that a path $\gamma : [0, 1] \to \overline{\Sigma}(X)$ (or by abuse its image) is *taut* if $\gamma^{-1}(L)$ is connected for every leaf L of $\overline{\mathcal{F}}$. If $A \subset \overline{\Sigma}(X)$, let $\mathrm{sat}(A)$ be the union of all leaves meeting A.

LEMMA 3.3. *If a path γ between $x, y \in \overline{\Sigma}(X)$ is taut, then $\|\gamma\| = d_{\overline{\mathcal{F}}}(x, y)$. If γ' is another path between x and y, then γ is contained in* $\mathrm{sat}(\gamma')$.

Proof. Let γ, γ' be two paths from x to y, with γ taut. We show $\|\gamma\| \leq \|\gamma'\|$ and $\gamma \subset \mathrm{sat}(\gamma')$. We may assume that both paths are in a general position with respect to $\overline{\mathcal{F}}$.

Since $\pi_1(\overline{\Sigma}(X)) = \overline{\mathcal{L}}$, there is a compact planar surface P and a map $f : P \to \overline{\Sigma}(X)$ sending one boundary component of P onto $\gamma\gamma'^{-1}$ and the others into leaves of $\overline{\mathcal{F}}$. Assume f is in a general position with respect to $\overline{\mathcal{F}}$.

We then get a measured foliation with finitely many isolated singularities in P, such that boundary components are contained in leaves, except one that consists of two arcs α, α' mapped by f to γ, γ'.

A non-singular leaf starting at a point of α has to reach the boundary of P: it cannot accumulate inside P, because of the transverse measure (Poincaré recurrence theorem, see [**FLP**, exposé 5], Théorème I.5). It cannot return to α because of tautness. Thus it has to reach α'. The same thing holds for the (finitely many) singular leaves. □

Using the absence of reflection, we now show:

LEMMA 3.4. *No leaf of $\overline{\mathcal{F}}$ meets the same component of $\pi^{-1}(D)$ twice: every component of $\pi^{-1}(D)$ is taut.*

Proof. Otherwise there would be a measured foliation on a surface P as before, the only difference being that the exceptional boundary component of P now consists of an arc β contained in a leaf and an arc β' transverse to the foliation. This implies that every regular leaf meeting β' is a segment with endpoints in β'. Since P is compact, any such regular leaf belongs to a maximal band of regular leaves joining two open subintervals (a, b) and (c, d) of β'. The leaves through the endpoints of these subintervals are singular. Since P is planar and the number of singular leaves is finite, there is a band of regular leaves joining two open subintervals (a, b) and (c, d) of β' having a common endpoint $b = c$. This point must be the center of a reflection. □

Let $A_1, \dots, A_n, \dots$ be the components of $\pi^{-1}(D)$. Let I_n be the intersection of A_n with $B_{n-1} = \mathrm{sat}(A_1 \cup \dots \cup A_{n-1})$. Since $\overline{\Sigma}(X)$ is connected, we may assume that I_n is non-empty for $n \geq 2$. The set B_n is then path-connected for every n.

LEMMA 3.5. *The set I_n is a closed connected subset of A_n.*

Connectedness is analogous to Lemma 1.7 in Rimlinger [**Rim1**]. Closedness is a consequence of segment closure (Proposition 2.2).

Proof. If $x, y \in I_n$, the interval $[x, y] \subset A_n$ is taut by Lemma 3.4. By Lemma 3.3, it is contained in B_{n-1} because B_{n-1} is saturated and path-connected. This shows that I_n is connected. We now show that it is closed.

If not, there exists $\eta > 0$ and an isometric map $\overline{p} : [0, \eta] \to A_n$ such that $\overline{p}(t) \in I_n$ if and only if $t > 0$. Choose a decreasing sequence t_p converging to 0 such that the leaf of $\overline{p}(t_p)$ contains a point q_p belonging to a fixed component A of $(A_1 \cup \ldots \cup A_{n-1})$. By Lemmas 3.3 and 3.4 there exists $\eta' < \eta$ and an isometric map $\overline{q} : [0, \eta'] \to A$ such that $\overline{q}(t)$ is in the same leaf as $\overline{p}(t)$ for $t > 0$ and $q_p = \overline{q}(t_p)$.

Using the covering projection from $\overline{\Sigma}(X)$ to $\Sigma(X)$, one gets maps $p, q : [0, \eta'] \to D$ such that $p(t)$ and $q(t)$ are in the same leaf of $\mathcal{F}$ for all t in $(0, \eta']$.

By Theorem 2.3, for some $0 < \eta'' \leq \eta'$, there is a 'band' of leaves from $p([0, \eta''])$ to $q([0, \eta''])$ joining $p(t)$ to $q(t)$. This band of leaves may be lifted to a band of leaves starting from $\overline{p}([0, \eta])$. Since the covering $\overline{\Sigma}(X)$ is defined by $\overline{\mathcal{L}}$, the leaf issued from $\overline{p}(t)$ in this lifted band terminates at $\overline{q}(t)$ for every t in $(0, \eta]$. By continuity, $\overline{p}(0)$ and $\overline{q}(0)$ belong to the same leaf. □

LEMMA 3.6. *Two points x, y in $A_1 \cup \ldots \cup A_n$ may be joined by a taut path γ contained in $B_n = \mathrm{sat}(A_1 \cup \ldots \cup A_n)$.*

Proof. We argue by induction on n, noting that the result is true for $n = 1$ by Lemma 3.4. It is enough to consider the case when $x \in A_n \setminus I_n$ and $y \notin A_n$. Let x' be the point of I_n closest to x on A_n (it exists because I_n is a non-empty closed subinterval). Choose $y' \in A_1 \cup \ldots \cup A_{n-1}$ in the same leaf as x'. The required path γ between x and y is obtained by concatenating the arc $[x, x'] \subset A_n$, a segment of a leaf between x' and y', and a taut path γ' between y' and y given by the induction hypothesis. The path γ is taut because the leaves meeting γ' meet $A_1 \cup \ldots \cup A_{n-1}$ while those meeting $[x, x')$ do not. □

Since every leaf of $\overline{\mathcal{F}}$ meets $\pi^{-1}(D)$, assertion (1) above follows from Lemmas 3.6 and 3.3.

To prove assertion (2), we can either apply Corollary III.5 from **[Lev3]** or argue as follows. We prove by induction that the metric space K_n associated with $(B_n, d_{\overline{\mathcal{F}}}\,|_{B_n})$ is a finite metric tree. This is true for $n = 1$ by Lemmas 3.4 and 3.3. From the way the taut path γ was constructed in the proof of Lemma 3.6, we see that one passes from K_n to K_{n+1} by isometrically glueing A_{n+1} along the closed subinterval I_{n+1}, so that K_{n+1} is indeed a finite tree. The space $T(X)$ is an $\mathbb{R}$-tree because it is the increasing union of the finite trees K_n. □

4. *Approximating free actions on $\mathbb{R}$-trees*

A *finite* $\mathbb{R}$-*tree* is a compact $\mathbb{R}$-tree which is the union of finitely many segments. Let G be a finitely generated group acting on an $\mathbb{R}$-tree T, and $\{g_1, \ldots, g_p\}$ a fixed system of generators for G. Let K be a finite subtree of T.

The elements $g_1, \ldots, g_p$ of Γ define partial isometries of K:

$$\psi_i : A_i = K \cap g_i^{-1}(K) \to B_i = g_i(K) \cap K$$

with $\psi_i(t) = g_i(t)$. The domains A_i and ranges B_i are finite subtrees. We assume that K is big enough for them to be non-empty. Let $\mathcal{K} = (K, \{\psi_i\})$.

From $\mathcal{K}$ we could obtain a system of isometries $X(K)$ on a multi-interval as in [**GLP1**, §2]. To approximate free actions, it turns out to be easier to work with finite $\mathbb{R}$-trees. Most definitions given for multi-intervals extend readily to $\mathcal{K}$.

In particular, we asssociate to $\mathcal{K}$ a foliated 2-complex $\Sigma = \Sigma(\mathcal{K})$ by glueing strips $A_i \times [0, 1]$ to K. We define $G(\mathcal{K}) = \pi_1\Sigma/\overline{\mathcal{L}}$ by killing loops contained in leaves.

It is easy to see that $G(\mathcal{K})$ is generated by $\{\overline{g_1}, \ldots, \overline{g_p}\}$, relations being words $\overline{g_{i_1}}^{\epsilon_1} \ldots \overline{g_{i_n}}^{\epsilon_n}$ such that there exists $x \in K$ with $\psi_{i_m}^{\epsilon_m} \ldots \psi_{i_n}^{\epsilon_n}(x) \in K$ for $1 \leq m \leq n$ and $\psi_{i_1}^{\epsilon_1} \ldots \psi_{i_n}^{\epsilon_n}(x) = x$.

We say that $\mathcal{K}$ has a *reflection* if there is a segment $s : [-\epsilon, +\epsilon] \to K$ and a $\mathcal{K}$-word fixing $s(0)$ and sending $s(-\epsilon)$ to $s(\epsilon)$.

We associate as before a metric space $T(\mathcal{K})$ to $\mathcal{K}$ by considering the natural pseudo-distance $d_{\overline{\mathcal{F}}}$ on the covering $\overline{\Sigma}$ of Σ associated with $\overline{L}$. We let $\pi : \overline{\Sigma} \to \Sigma$ be the covering map.

THEOREM 4.1. *Let $\mathcal{K} = (K, \{\psi_i\})$ be a finite set of partial isometries between closed subtrees of a finite $\mathbb{R}$-tree K. If $\mathcal{K}$ is without reflection, then $T(\mathcal{K})$ is an $\mathbb{R}$-tree with a free action of $G(\mathcal{K})$. Furthermore, for any component $\overline{K}$ of $\pi^{-1}(K)$, the restriction of π to $\overline{K}$ is an isometry from $(\overline{K}, d_{\overline{\mathcal{F}}})$ onto K.*

Proof. The proof is the same as for Theorem 3.2. In the course of the proof, corresponding to Lemma 3.4, one shows that no leaf of $\overline{\mathcal{F}}$ meets the same component of $\pi^{-1}(K)$ twice. It follows that $(\overline{K}, d_{\overline{\mathcal{F}}})$ is isometric to K for every component $\overline{K}$ of $\pi^{-1}(K)$. To prove the equivalent of Lemma 3.5, we note that Theorem 2.3 also holds on K. □

From now on, we assume that G is a finitely generated group acting freely on an $\mathbb{R}$-tree T. Let K be a finite subtree of T such that each A_i is non-empty (see above). Since G acts freely, it is clear that $\mathcal{K}$ has no reflection.

From the presentation of $G(\mathcal{K})$ given above we get a natural epimorphism $G(\mathcal{K}) \to G$. Since G is finitely presented (see [**GLP1**]), this epimorphism is an isomorphism for K big enough. In what follows, we identify G and $G(\mathcal{K})$.

Hence if K is a sufficiently large finite subtree of T, Theorem 4.1 yields a free action of the group G on the $\mathbb{R}$-tree $T(\mathcal{K})$. We will fix a component $\overline{K}$ of $\pi^{-1}(K)$. Using the 'furthermore' in Theorem 4.1, we may then embed K isometrically into $T(\mathcal{K})$, by identifying K with the image of $\overline{K}$ in $T(\mathcal{K})$.

It is not always true that $T(\mathcal{K})$ is (equivariantly) isometric to T for K big enough. Indeed, if G is a free group, it happens precisely when the action on T is *geometric* (see [**GL**]). But our goal (Theorem 4.3) is to show that $T(\mathcal{K})$ is a strong approximation of T.

Recall that an action of a group on an $\mathbb{R}$-tree is said to be *minimal* if there is no proper invariant subtree. If $g \in G$, the *translation length* of g on T is $\ell_T(g) = \inf\{d(x, gx)/x \in T\}$. Also recall the following definition from [**MO**].

Definition 4.2. Let T, T' be $\mathbb{R}$-trees endowed with an action of a group G. A morphism from T to T' is an equivariant continuous map $f : T \to T'$ such every segment in T may be subdivided into finitely many subsegments that inject isometrically in T'.

A sequence of $\mathbb{R}$-trees T_n with an action of G *converges strongly* (see **[GS]**) to an $\mathbb{R}$-tree T with a minimal action of G if there exist morphisms f_n from T_n to T and $f_{k,n}$ from T_k to T_n for $k < n$, such that $f_k = f_n \circ f_{k,n}$, and for every x, y in T_k there exists some $n \geq k$ such that $d(f_{k,n}(x), f_{k,n}(y)) = d(f_k(x), f_k(y))$. In particular, for every $g \in G$, $\ell_{T_n}(g)$ is eventually constant and equal to $\ell_T(g)$ (see **[GS]**).

A sequence of metric G-spaces Y_n converges to a metric G-space Y for the *equivariant Gromov topology* if, given a finite subset K of Y, a finite subset P of G, and $\epsilon > 0$, then for n big enough, there is a map $\rho_n : K \to Y_n$ such that for every x, y in K and g in P,

$$|d(gx, y) - d(g\rho_n(x), \rho_n(y))| \leq \epsilon.$$

For further information on this topology, see **[Pau1]** and **[Pau2]**, where a slight mistake in the definition took place, as pointed out to the third author by Skora, all results remaining true.

It is proved in **[Pau1]** that, for $\mathbb{R}$-trees, convergence for the equivariant Gromov topology implies convergence of the translation lengths, the converse being true if the action is minimal and if the translation lengths are not the absolute values of a homomorphism from G to $\mathbb{R}$. The heuristic difference between strong convergence and equivariant Gromov convergence of $\mathbb{R}$-trees T_n towards T is the following: if the convergence is strong, one is able to lift finite subsets from T to T_n isometrically, while otherwise one may lift finite subsets only isometrically up to ϵ.

THEOREM 4.3. *If G is a finitely generated group acting freely and minimally on an $\mathbb{R}$-tree T, then $\mathcal{K}$ is without reflection for every finite subtree K in T. If K_n is an increasing sequence of finite subtrees with union T, then $G(\mathcal{K}_n)$ is isomorphic to G for n big enough and $T(\mathcal{K}_n)$ converges to T strongly, hence also in the equivariant Gromov topology.*

Proof. Let $K \subset K'$ be finite subtrees of T. Recall that we have chosen a component $\overline{K}$ of $\pi^{-1}(K)$, and thus embedded K isometrically into $T(\mathcal{K})$ (and similarly for K'). The natural inclusion $\Sigma(\mathcal{K}) \to \Sigma(\mathcal{K}')$ is a homotopy equivalence. We lift it to a map $\overline{\Sigma}(\mathcal{K}) \to \overline{\Sigma}(\mathcal{K}')$, sending $\overline{K}$ to $\overline{K}'$, which does not increase (pseudo)-distances. The induced map $T(\mathcal{K}) \to T(\mathcal{K}')$ is a morphism, since its restriction to K is isometric and every segment in $T(\mathcal{K})$ may be covered by finitely many images $h_i K$, $h_i \in G$. Similarly, there are natural morphisms $T(\mathcal{K}) \to T$ inducing the identity on K. Since every segment of T is contained in K_n for n big enough, strong convergence follows. □

Remark. There is another possible approach to Theorem 4.3, using Lemmas 3.3 and 3.4 as well as Corollary III.5 from **[Lev3]** but not Theorem 2.3. It is based on the fact that the length function of the action of G on $T(\mathcal{K})$ is greater than or equal to the length function of the action of G on T, so that the action of G on $T(\mathcal{K})$ is *free*.

There is a combinatorial description of the $\mathbb{R}$-tree $T(\mathcal{K})$, due to Rips. On $K \times G(\mathcal{K})$, define a relation $\sim$ in the following way. First, for x, y in K and g, h in $G(\mathcal{K})$, say that (x, g) is in relation with (y, h) if (with the above notations) there exists i in $\{1, \ldots, p\}$

such that $y = \psi_i(x)$ and $g = h\overline{g_i}$ (equality in $G(\mathcal{K})$, with the presentation given above). We define $\sim$ as the equivalence relation generated by the above relation. Then

$$T(\mathcal{K}) = (K \times G(\mathcal{K}))/_{\sim}.$$

In these terms, the action of $G(\mathcal{K})$ on $T(\mathcal{K})$ is induced by the action on $K \times G(\mathcal{K})$ defined by $g \cdot (x, h) = (x, gh)$.

The distance on $T(\mathcal{K})$ is induced by the natural pseudodistance $\overline{\delta}$ on $K \times G(\mathcal{K})$ defined as follows (see [**GL**] in the free group case):

$$\overline{\delta}((x, g), (y, h)) = \inf\{d_K(x, x_k) + d_K(\psi_{i_k}(x_k), x_{k-1}) + \cdots + d_K(\psi_{i_1}(x_1), y)\}$$

where d_K is the distance in K and the infimum is taken over all words $\overline{g_{i_1}} \cdots \overline{g_{i_k}}$ representing $h^{-1}g$ in $G(\mathcal{K})$ and all points x_j in the domain of ψ_{i_j}.

Remark. It is possible to approximate any action of a finitely generated group on an $\mathbb{R}$-tree by actions suitably associated to finite sets of partial isometries on finite trees (see [**LP**]).

5. *Closed pseudogroups*

The following notion is a variation on the notion of pseudogroup, as studied by Lie and Cartan (see for instance [**SS**]), Veblen–Whitehead [**VW**], Ehresmann [**Ehr**], Haefliger [**Hae1**], [**Hae2**], Molino [**Mol**], Salem [**Sal**].

Definition 5.1. A closed pseudogroup on a multi-interval D is a set $\mathcal{P}$ of isometries between closed subintervals of D such that:

(1) the identity on every component of D belongs to $\mathcal{P}$,
(2) (inversion) if $\varphi \in \mathcal{P}$, then $\varphi^{-1} \in \mathcal{P}$,
(3) (composition) for every φ, φ' in $\mathcal{P}$ then $\varphi \circ \varphi'$ belongs to $\mathcal{P}$,
(4) (restriction) if φ belongs to $\mathcal{P}$ and A is a closed subinterval of the domain of φ, then $\varphi|_A : A \to \varphi(A)$ belongs to $\mathcal{P}$,
(5) (extension) for every $\varphi : A \to B, \varphi' : A' \to B'$ in $\mathcal{P}$ with $A \cap A'$ non-empty, if there is an isometry $\varphi'' : A \cup A' \to B \cup B'$ whose restriction to A, A' is respectively φ, φ', then φ'' belongs to $\mathcal{P}$.

Note that the definition makes sense for every metric space D (for instance an $\mathbb{R}$-tree), using connected subsets instead of subintervals. We allow singletons in a closed pseudogroup, and restrictions to points in condition (4) of the definition. The composition of two partial isometries is understood to be the maximally defined one (maybe the empty map).

An *open pseudogroup* (or pseudogroup for short) is defined as a closed pseudogroup, replacing closed by open and allowing extensions with any number of elements. We have to take finitely many of them in the closed case to be sure that the union of domains is still closed.

Note that the intersection of any family of closed (resp. open) pseudogroups is again a closed (resp. open) pseudogroup. For instance, the set of all isometries between closed (resp. open) subintervals of a multi-interval is a closed (resp. open) pseudogroup.

Definition 5.2. The closed pseudogroup $\mathcal{P}(X)$ generated by a system of isometries $X = (D, \{\varphi_j\}_{j=1\cdots k})$ on a multi-interval D is the intersection of all closed pseudogroups on D containing every φ_j. A closed pseudogroup is said to be finitely generated if it is generated by a system of isometries.

Note that the elements of the closed pseudogroup generated by X are restrictions of extensions of X-words.

Main example. The main examples of finitely generated closed pseudogroups are obtained by taking a compact manifold with a measured foliation (maybe with isolated singularities, such as thorns, centers, multi-saddles or Morse type saddles) or a finite foliated n-complex (same type of singularities), by taking D to be a set of closed transversal arcs meeting every leaf, and by taking $\mathcal{P}$ to be the holonomy closed pseudogroup. An easy compactness argument using flow boxes shows that D may be chosen to be compact and that $\mathcal{P}$ is finitely generated. For open pseudogroups, and measured foliations without singularities, this was proved by Sacksteder [**Sac**]. The suspension construction $(\Sigma(X), \mathcal{F})$ in §1 shows that every system of isometries on a multi-interval may be obtained that way.

If $\mathcal{P}$ is a closed (resp. open) pseudogroup on D, then the orbits of $\mathcal{P}$ are the equivalence classes for the equivalence relation defined on D by

$$x \sim y \iff \exists(\varphi : A \to B) \in \mathcal{P},\quad x \in A, y \in B \text{ and } y = \varphi(x).$$

If X is a system of isometries, the orbits of X (resp. $\mathring{X}$) are the orbits of the closed (resp. open) pseudogroup generated by X (resp. $\mathring{X}$). Note that the open pseudogroup $\mathcal{P}(\mathring{X})$ generated by $\mathring{X}$ is not necessarily the set of restrictions of elements of $\mathcal{P}(X)$ to the interiors of their domains.

The following definition is an analogue for closed pseudogroups of the notion of equivalent pseudogroups developed by Haefliger [**Hae1**], [**Sal**].

Definition 5.3. Let $\mathcal{P}, \mathcal{P}'$ be closed pseudogroups on multi-intervals D, D'. An equivalence between $\mathcal{P}, \mathcal{P}'$ is a finite set $\mathcal{Q} = \{\varphi_i : A_i \to B_i\}_{i=1\cdots m}$ of partial isometries from closed intervals in D onto closed intervals in D', satisfying the following conditions:

- The A_i's (resp. B_i's) cover D (resp. D').
- $\mathcal{Q} \circ \mathcal{P} \circ \mathcal{Q}^{-1} \subset \mathcal{P}'$ and conversely, that is for every φ_i, φ_j in $\mathcal{Q}$, for every φ in $\mathcal{P}$ and φ' in $\mathcal{P}'$, we have $\varphi_i \circ \varphi \circ \varphi_j^{-1}$ belongs to $\mathcal{P}'$, and $\varphi_i^{-1} \circ \varphi' \circ \varphi_j$ belongs to $\mathcal{P}$.

An equivalence between two systems of isometries is an equivalence between the closed pseudogroups they generate.

It is difficult to relax the finiteness assumption in the definition of an equivalence, since if we allow uncountably many elements in an equivalence, any two closed pseudogroups with the same orbit space cardinality would be equivalent.

Main example. The main example of equivalence $\mathcal{Q}$ is, as in Haefliger's case, obtained by changing the complete transversal D in the main example above. The fact that $\mathcal{Q}$ is finite follows as usual by a compactness argument on flow boxes.

Rips has introduced examples of equivalences (that he called 'elementary moves'). We have needed in [**GLP1**] only two of them, 'splitting' and 'erasing an interval covered once' (in [**GLP1**, Proposition 3.5 and §7]). We recall the first one since we have slightly generalized it to remove the assumption of pureness.

Splitting. Let X be a system of isometries. Let x be an interior point of a component I of D. We first split the bases A_j (resp. B_j) containing x in their interior, and split B_j (resp. A_j) at the corresponding point $\varphi_j(x)$ (resp. $\varphi_j^{-1}(x)$). We replace φ_j by two isometries, the restrictions of φ_j to the closures of the two components of $A_j - \{x\}$ (resp. $A_j - \{\varphi_j^{-1}(x)\}$). We then replace I by two disjoint intervals I_1, I_2, isometric to the closures of the components of $I - \{x\}$, embedded in $\mathbb{R}$ disjointly from the other components of D, so that x is replaced by two points x_1, x_2. Let D' be the new multi-interval thus obtained. Each isometry in X gives rise to a partial isometry of D' in the obvious way, by transferring the bases that were above I either to I_1 or I_2 accordingly. If $\{x\}$ was the base of a singleton, then transfer it arbitrarily to either x_1 or x_2. The new system of isometries X' on D' is then obtained by taking the partial isometries defined in this manner, and adding a singleton taking x_1 to x_2.

It is easy to see that X and X' are equivalent.

Theorem 2.3 has important applications to finitely generated closed pseudogroups.

PROPOSITION 5.4. (1) *Let $\mathcal{P}$ be a finitely generated closed pseudogroup on a multi-interval D. Then $\mathcal{P}$ is closed under (possibly infinite) extensions: if $\phi : A \to B$ is an isometry between closed intervals of D, if $(\phi_\alpha : A_\alpha \to B_\alpha)_\alpha$ is any family in $\mathcal{P}$, with $\mathring{A} \subset \cup_\alpha A_\alpha$ and $\phi_{|A_\alpha \cap A} = \phi_{\alpha|A_\alpha\cap A}$, then ϕ belongs to $\mathcal{P}$.*

(2) *The equivalence relation whose classes are the orbits of $\mathcal{P}$ is segment closed.*

Furthermore:

PROPOSITION 5.5. *If $\mathcal{P}$ and $\mathcal{P}'$ are finitely generated closed pseudogroups on a multi-interval D such that the $\mathcal{P}$-orbit and the $\mathcal{P}'$-orbit of x are equal for every x in D, then $\mathcal{P}$ and $\mathcal{P}'$ are equal.*

Proof. Any singleton of $\mathcal{P}$ is a singleton of $\mathcal{P}'$ and conversely. Furthermore, any element of $\mathcal{P}$ with domain of non-empty interior is in $\mathcal{P}'$ and conversely, by Theorem 2.3. □

If $\mathcal{P}$ is a closed (resp. open) pseudogroup on D, and D' is a multi-interval contained in D, then the set of restrictions of elements of $\mathcal{P}$ to D' (resp. the interior of D') is a closed (resp. open) pseudogroup called the restriction of $\mathcal{P}$ to D'. For instance, any subgroup G of Isom($\mathbb{R}$) canonically defines a closed (resp. open) pseudogroup on a multi-interval D, simply by taking the set of restrictions of elements of G to any closed (resp. open) interval of D. Any *homogeneous* system of isometries (in the sense of [**GLP1**, §4]) may be obtained in this way (up to changing the embedding of D into $\mathbb{R}$). This fact is a consequence of Proposition 5.5.

Theorem 2.3 also implies that the quasi-isometry type of the orbits of a system of isometries depends only on the generated pseudogroup. Recall that orbits of X may be viewed as Cayley graphs (see §1). We consider them as metric spaces, by giving length 1 to every edge.

A (λ, μ)-quasi-isometry between metric spaces E, E' is a map $f : E \to E'$ such that

$$\forall x, y \in E, \ \ \frac{1}{\lambda} d(x, y) - \mu \leq d(f(x), f(y)) \leq \lambda d(x, y) + \mu$$

and there is some $C \geq 0$ such that $d(x', f(E)) \leq C$ for all x' in E'. Note that the composition of a (λ, μ)-quasi-isometry and a (λ', μ')-quasi-isometry is a $(\lambda\lambda', \lambda\mu' + \mu)$-quasi-isometry.

PROPOSITION 5.6. *Let X, X' be systems of isometries on a multi-interval D, having the same orbits. Then for every x in D, the X-orbit and X'-orbit of x are quasi-isometric.*

Proof. By Theorem 2.3, every generator of X (resp. X') has a finite expression in terms of the generators of X' (resp. X). We may then apply the classical argument (see for instance **[Gro]**) used to show that the quasi-isometry type of the Cayley graph of a finitely generated group does not depend of the choice of a finite generating system. □

More generally:

PROPOSITION 5.7. *Let X, X' be system of isometries on multi-intervals D, D'. If $\mathcal{Q}$ is an equivalence between X and X', then for every x in D the X-orbit of x and X'-orbit of $\mathcal{Q}(x)$ are quasi-isometric.*

Note that if $x \in D$ belongs to the domain of ψ, ψ' in $\mathcal{Q}$, then the X'-orbits of $\psi(x), \psi'(x)$ coincide, and conversely when permuting X and X'.

Proof. For every φ in $\mathcal{P}(X)$ which is either a generator of X or the identity of some component of D and every ψ, ψ' in $\mathcal{Q}$, by definition of the generated closed pseudogroup, the domain of $\psi' \circ \varphi \circ \psi^{-1}$ may be finitely subdivided so that the restriction of $\psi' \circ \varphi \circ \psi^{-1}$ to every piece is a restriction of some X'-word. Define λ to be the maximum of the lengths of these X'-words for all such $\psi' \circ \varphi \circ \psi^{-1}$ (with non empty domain) and of the lengths of the X-words obtained similarly by permuting X and X'.

It is clear that any map from the vertices of the X-orbit of x to the vertices of the X'-orbit of $\mathcal{Q}(x)$, which to y associates any point of $\mathcal{Q}(y)$ is (λ, λ)-quasi-isometric, hence that these orbits are $(\lambda, 2\lambda + 1)$-quasi-isometric. □

We now show that the group $G(X)$ associated with a system of isometries X, and the $\mathbb{R}$-tree $T(X)$ constructed in §3, depend only on the equivalence class of X.

LEMMA 5.8. *If X, X' are systems of isometries, such that X is obtained from X' by adding a new generator $\phi : A \to B$ such that x and $\phi(x)$ are in the same X'-orbit for every x in the domain of ϕ, then $G(X)$ and $G(X')$ are isomorphic.*

Proof. Fix a base point in $A \subset \Sigma(X') \subset \Sigma(X)$. Consider the composite map $\pi_1\Sigma(X') \to \pi_1\Sigma(X) \to G(X)$ induced by inclusion and canonical projection. Any closed loop in a leaf of $\Sigma(X')$ yields a closed loop in a leaf of $\Sigma(X)$, hence one has a map $G(X') \to G(X)$. Any path γ in $\pi_1\Sigma(X)$ is homotopic (rel. base point) to a path meeting the interior $A \times (0, 1)$ of the new strip in finitely many leaves $\{x_j\} \times (0, 1)$, every such leaf being covered only once. We will assume our paths to have this property.

Since x_j and $\phi(x_j)$ are in the same leaf of $\Sigma(X')$, the image of γ in $G(X)$ is equal to the image of a path contained in $\Sigma(X')$. So the map $G(X') \to G(X)$ is onto.

If the homotopy class of a path γ in $\pi_1\Sigma(X')$ is trivial in $G(X)$, then γ is homotopic (rel. base point) in $\Sigma(X)$ to a composite path $\Pi_{i=1}^{m}\alpha_i^{-1}\beta_i\alpha_i$ with β_i a closed loop in a leaf of $\Sigma(X)$. Take a disk D in a general position with respect to $A \times \{\frac{1}{2}\}$, with boundary $\gamma^{-1}\Pi_{i=1}^{m}\alpha_i^{-1}\beta_i\alpha_i$.

The intersection $D \cap (A \times \{\frac{1}{2}\})$ consists of finitely many arcs and loops. We may remove them one by one by considering at each step the outermost disk. We get that γ is homotopic (rel. base point) in $\Sigma(X)$ to a composite path $\Pi_{i=1}^{m'}\alpha'^{-1}_i\beta'_i\alpha'_i$ with β'_i and α'_i contained in $\Sigma(X')$.

We want to prove that γ is, in fact, homotopic (rel. base point) in $\Sigma(X')$ to such a product path (hence trivial in $G(X')$). Subdividing, we see that it is enough to prove it for a loop of the form $[x, \phi(x)] \cup [\phi(x), \phi(y)] \cup [\phi(y), y] \cup [x, y]$ where $[x, y] \subset A$ and $[x, \phi(x)], [\phi(y), y]$ are arcs contained in leaves of $\Sigma(X')$ (such a loop is nullhomotopic in $\Sigma(X)$). But this follows from Theorem 2.3, as after finitely many subdivisions, one may assume that $[x, y]$ is contained in a band of leaves in $\Sigma(X')$ between $[x, y]$ and $[\phi(x), \phi(y)]$. □

Note that this cannot be generalized to the non-finitely generated case. For example, consider $\{\phi_i\}_{i\in\mathbb{N}}$ where ϕ_0 is the singleton $\{0\} \to \{2\}$ and $\phi_n : [\frac{1}{n+1}, \frac{1}{n}] \to [2+\frac{1}{n+1}, 2+\frac{1}{n}]$ for every $n \geq 1$ is the translation by 2. Then the group naturally associated to this infinite system of isometries is the free group of rank 2. But if we add the positive generator $[0, 1] \to [2, 3]$, the associated group becomes $\mathbb{Z}$.

PROPOSITION 5.9. *If two systems of isometries X_1, X_2 on multi-intervals D_1, D_2 are equivalent, then the groups $G(X_1)$, $G(X_2)$ associated with X_1, X_2 are isomorphic.*

Proof. We first claim that a splitting does not change the group. This may be found in [**GLP1**], but we give the argument for the sake of completeness. Suppose X' is a system of isometries obtained from a system of isometries X by a splitting. When we split one base, then we replace one band by two bands, hence introducing a new generator in the group. But one creates a new loop in leaves (going by the left side of one band and coming back by the right side of the other), giving a relation that kills the new generator. When we split D, we immediately introduce a new singleton, hence a new band (reduced to one leaf). Pinching this leaf induces a homotopy equivalence between the old and new foliated 2-complexes, preserving the loops in leaves.

Let $\mathcal{Q}$ be an equivalence between $\mathcal{P}(X_1)$ and $\mathcal{P}(X_2)$. Split X_2 at all endpoints of ranges of elements of $\mathcal{Q}$. One gets a new equivalence by taking the appropriate restrictions of the elements of $\mathcal{Q}$. Keeping the same notations, the range of any component of $\mathcal{Q}$ is now a component of D_2.

Assume $\psi: A \to B$, $\psi': A' \to B$ are elements of $\mathcal{Q}$ with the same range. We claim that we may replace X_1 by a system of isometries having the same associated group, equivalent to X_2 by an equivalence having one less element.

Let E be the union of domains of elements of $\mathcal{Q}$ different from ψ. If one replaces ψ by its (finitely many) restrictions to every component of $\overline{A - E}$, one still gets an equivalence between X_1 and X_2, since if ψ'' in $\mathcal{Q}$ coincides with ψ on $I \subset A$, then $\psi \circ (\psi'')^{-1}$ defined on $\psi''(I)$ is in $\mathcal{P}(X_2)$. Split X_1 at the endpoints of every component J of $\overline{A - E}$. This does not change the equivalence, since no such endpoint is in the interior of a domain of an element of $\mathcal{Q}$. Let J again be the corresponding component of D_1. For any such J, add to X_1 a generator ϕ_J sending J onto $(\psi')^{-1} \circ \psi(J)$ by

$(\psi')^{-1} \circ \psi$. Since ϕ_J is in $\mathcal{P}(X_1)$, this does not change the group by Lemma 5.8. Pinch to a point each leaf of the band $J \times [0, 1]$ in $\Sigma(X_1)$ corresponding to each ϕ_J. Since J was a component of D, we obtain a new system of isometries. This leaf-wise homotopy equivalence does not change the associated group. The elements of $\mathcal{Q}$ corresponding to ψ are no longer necessary to the equivalence, since they now coincide with a restriction of ψ'. This proves the claim.

By induction, we may assume that the ranges of two elements of $\mathcal{Q}$ are disjoint components of D_2. Split X_1 at all endpoints of domains of elements of $\mathcal{Q}$, and split open accordingly the elements of $\mathcal{Q}$. Split X_2 at all endpoints of ranges of elements of the new $\mathcal{Q}$ (that are in the interior of D_2). Since a component of the old D_2 contained precisely one range of the old $\mathcal{Q}$, the new $\mathcal{Q}$ without further splitting naturally defines an equivalence between the new X_1 and X_2. Now both domains and ranges of elements of $\mathcal{Q}$ are components of D_1, D_2 respectively.

Proceeding as above, one can change X_2, $\mathcal{Q}$ so that a component of D_1 contains precisely one domain of element of $\mathcal{Q}$ (keeping the analogous fact true for D_2). Hence the equivalence defines a homeomorphism $\psi: D_1 \to D_2$ which is an isometry on each component.

Now add to X_1 (resp. X_2) the images of X_2 (resp. X_1) transferred on D_1 (resp. D_2) by using the isometry ψ. By Lemma 5.8, the associated groups are unchanged. We now claim that the associated groups are isomorphic. Indeed ψ extends to a foliation-preserving homeomorphism between the foliated 2-complexes $\Sigma(X_1)$ and $\Sigma(X_2)$. □

PROPOSITION 5.10. *If X, X' are equivalent systems of isometries, then $T(X)$, $T(X')$ are isometric by an isometry which commutes with the actions of $G(X)$, $G(X')$.*

Proof. Indeed, the proof of Proposition 5.9 may be followed to get the result. □

Ideas in the proof of Proposition 5.9 will be used in [**Pau4**] to give a conceptual computation of the group associated to a homogeneous system (see [**GLP1**, Proposition 4.2]).

Finally, we discuss reflections in systems of isometries.

Recall that a *reflection* in a closed or open pseudogroup is an element with non-degenerate domain and derivative -1, having a fixed point, called its *center*. According to Theorem 2.3, a finitely generated closed pseudogroup $\mathcal{P}$ is without reflection if and only if there is no x such that $x + t$ is in the orbit of $x - t$ for $t > 0$ small.

Observe that being without reflection is not invariant under equivalence. For one thing, a splitting may destroy a reflection. Furthermore any closed pseudogroup is equivalent to one having a reflection. To see this, consider a tripod, i.e. the finite tree union of three unit intervals $[o, a][o, b]$, $[o, c]$ glued at o, and consider the partial isometry of the tripod sending $[a, b]$ to $[a, c]$. Then the natural systems of isometries obtained as in §1 by cutting the tripod into either $[a, b] \vee [o, c]$ (disjoint union) or $[b, c] \vee [a, o]$ generate equivalent closed pseudogroups, but the second one has a reflection. Moreover, they are equivalent to the identity on an interval of length 2, hence they may be included in any pseudogroup.

Even after finitely many splittings, we may still have reflections. For instance if a center of reflection x is in the interior of another generator of X, and if the orbits of $\mathring{X}$

are dense in $\mathring{D}$, there are infinitely many centers of reflections.

PROPOSITION 5.11. *If $\mathcal{P}$ is a finitely generated closed pseudogroup, then there are finitely many orbits of centers of reflections.*

Proof. Since E (as defined in [**GLP1**, §3]) is finite, and since the result is obvious for families of finite orbits, we need only consider the case when X has only one minimal component (by [**GLP1**, §3]). The homogeneous case is easy, since then the centers of reflections are the fixed points of elements of the finitely generated subgroup P of Isom($\mathbb{R}$). In the non-homogeneous case X_{-t} (as defined in [**GLP1**, §4]) has only a finite number of orbits of centers of reflection, since X_{-t} is a finite union of (possibly twisted) families of finite orbits (see [**GLP1**, §3]). Furthermore this finite number is bounded independently of t. Indeed, the number of disjoint Moebius bands that can be embedded in $\Sigma(X_{-t})$ is bounded by the rank of $\mathrm{Hom}(\pi_1\Sigma(X_{-t}), \mathbb{Z}/2\mathbb{Z})$, which is uniformly bounded since the homotopy type of $\Sigma(X_{-t})$ does not change. Since any finite number of orbits in X may be found in X_{-t} for t small enough, the result follows. □

In [**Gus**], there is a computation of the exact number of orbits in terms of the algebraic structure of a slight variation of the group $G(X)$.

Definition 5.12. A closed pseudogroup $\mathcal{P}$ on a multi-interval D is almost without reflection if only finitely many points of D are centers of reflections.

It is clear that any closed pseudogroup equivalent to $\mathcal{P}$ is almost without reflection. Indeed, after finitely many splittings, the centers of reflections that remain are in the interior of the domains of the equivalence, so that reflections may be transferred.

THEOREM 5.13. *The conclusions of Theorem 3.2 hold if X is a system of isometries almost without reflection.*

Proof. After a finite number of splittings, which does not change $G(X)$ nor $T(X)$ nor the fact that $d_{\mathcal{F}}(x, y) = 0$ if and only if x, y are in the same leaf, we may assume that no center of reflection of $\mathcal{P}(X)$ is in the interior of a component of the multi-interval D. Hence the first claim of the proof of Theorem 3.2 holds, and the rest of the proof is the same. □

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