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IRREDUCIBLE AUTOMORPHISMS OF F_n HAVE NORTH–SOUTH DYNAMICS ON COMPACTIFIED OUTER SPACE

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Abstract We show that if an automorphism of a non-abelian free group F_n is irreducible with irreducible powers, it acts on the boundary of Culler–Vogtmann’s outer space with north–south dynamics: there are two fixed points, one attracting and one repelling, and orbits accumulate only on these points. The main new tool we use is the equivariant assignment of a point $Q(X)$ to any end $X \in \partial F_n$, given an action of F_n on an \mathbf{R} -tree T with trivial arc stabilizers; this point $Q(X)$ may be in T , or in its metric completion, or in its boundary.

Keywords: free group; automorphism; outer space; tree; pseudo-Anosov; irreducible

AMS 2000 *Mathematics subject classification:* Primary 20F65; 20E05; 20E08
Secondary

1. Introduction

In [8], Culler and Vogtmann introduced a space of ‘moduli on marked graphs’. This space, CV_n , now also known under the name of *outer space*, is finite dimensional, contractible and, most importantly, its spine admits a discrete co-compact action with finite point stabilizers of $\text{Out } F_n$, the group of outer automorphisms of the non-abelian free group F_n (see the survey [22]). The emerging picture has a striking analogy with the action of the mapping class group Γ_g on the Teichmüller space \mathcal{T}_g —the only really major difference being that \mathcal{T}_g is a manifold, while CV_n is only a simplicial complex (with some faces of certain simplices missing).

In [2], Bestvina and Handel, inspired by Thurston’s work on surface homeomorphisms, introduced a new basic tool into the theory of free group automorphisms. They developed the notion of *train track maps*, and they used it to prove deep facts (like the Scott Conjecture) about automorphisms of F_n . One of their fundamental contributions is the introduction of *irreducible* automorphisms of F_n , an analogue of Thurston’s pseudo-Anosov homeomorphisms of a closed surface S_g of genus g . An automorphism $\alpha \in \text{Aut } F_n$

(or its image $\hat{\alpha} \in \text{Out } F_n$) is *irreducible with irreducible powers* (IWIP) if no proper free factor H of F_n is mapped by any positive power of α to a conjugate of H .

One of the crucial innovative contributions of Thurston to surface theory is his boundary $\partial\mathcal{T}_g$, which compactifies \mathcal{T}_g to give topologically a closed ball $\bar{\mathcal{T}}_g$ of dimension $6g - 6$. Using Skora's Theorem [21], we can now describe $\partial\mathcal{T}_g$ as the projectivized space $SLF(\pi_1 S_g)$ of small actions of $\pi_1 S_g$ on \mathbf{R} -trees. In one of the first papers on \mathbf{R} -trees [7], Culler and Morgan have shown that for any finitely generated group G the space $SLF(G)$ is compact.

Now, considering the special case $G = F_n$, and observing that CV_n embeds naturally into $SLF(F_n)$, it is natural to consider the closure $\overline{CV_n}$ of this image. It consists of projective classes $[T]$ of very small actions of F_n on \mathbf{R} -trees T (see [22]). The *Thurston boundary* ∂CV_n of CV_n is defined as the difference $\overline{CV_n} \setminus CV_n$. As in the mapping class group case, the action of $\text{Out } F_n$ on CV_n extends canonically to an action on $\overline{CV_n}$.

It is well known that the action of a pseudo-Anosov automorphism on $\bar{\mathcal{T}}_g$ has *north-south dynamics*: there are precisely two fixed points, both in $\partial\mathcal{T}_g$, one is repelling and the other is attracting, and every other orbit in $\bar{\mathcal{T}}_g$ has the attractor as forward and the repeller as backward limit. The main result of this paper is that the precise analogy is true for IWIP automorphisms of F_n . It answers a question asked in [4].

Theorem 1.1. *Let $\alpha \in \text{Aut } F_n$ be irreducible with irreducible powers. Its action on the closure $\overline{CV_n}$ of outer space has north-south dynamics: there exist $[T^+], [T^-] \in \partial CV_n$ such that $\alpha^p([T])$ converges (locally uniformly) to $[T^+]$ as $p \rightarrow +\infty$ for all $[T] \neq [T^-]$, and $\alpha^{-p}([T]) \rightarrow [T^-]$ for all $[T] \neq [T^+]$.*

It was known beforehand (see [1, 14, 18, 20]) that every $\alpha \in \text{Aut } F_n$ has at least one fixed point in $\overline{CV_n}$. It has been shown in [15, 16] that IWIP automorphisms have precisely two fixed points, both in ∂CV_n , and in [4, 13] that the orbits in the ‘interior’ CV_n all do converge from the repeller to the attractor. The analogous result for orbits on ∂CV_n , however, turns out to be quite difficult. The main new tool we use is the equivariant assignment of a point $Q(X)$ to any end $X \in \partial F_n$, given an action of F_n on an \mathbf{R} -tree T with trivial arc stabilizers; this point $Q(X)$ may be in T , or in its metric completion \bar{T} , or in the boundary of T (see §3).

One may wonder about the action of an arbitrary automorphism on ∂CV_n . See [3, 6] for dynamics of polynomially growing automorphisms. It is tempting to conjecture that $\text{Out } F_n$ acts on ∂CV_n with *uniformly finite limit sets*: there exists a constant K (depending only on n) such that, for every $\alpha \in \text{Aut } F_n$ and every $[T] \in \partial CV_n$, the sequence $\alpha^p([T])$ has at most K limit points. The analogous statement for the action of Γ_g on $\partial\mathcal{T}_g$ follows easily from Nielsen–Thurston theory (a detailed exposition appears in [17]).

In §§2 and 3 we consider an arbitrary \mathbf{R} -tree T with dense orbits and trivial arc stabilizers. We show (corollary 2.3) that there exist F_n -equivariant maps $f : T_0 \rightarrow T$, with T_0 a free simplicial F_n -tree, having arbitrarily small backtracking. This allows us in §3 to associate a point $Q(X)$ to every end $X \in \partial F_n$. These constructions are quite general and may have applications elsewhere.

In §4 we recall basic facts about train tracks, laminations, trees. In §5 we prove that there exists a leaf of one of the two α -invariant laminations whose ends X, X' satisfy $Q(X) \neq Q(X')$. In §6 we deduce the main theorem from this key fact, using a convergence criterion due to Bestvina *et al.* [4]. For the convenience of the reader, we provide a proof of this criterion in §7, with more details than in [4].

2. Maps with small backtracking

Let T_0 be a simplicial \mathbf{R} -tree with a free isometric action of F_n . We denote $\pi : T_0 \rightarrow T_0/F_n$ the quotient map and we let $v(T_0)$ be the total length of the graph T_0/F_n .

Let T be an \mathbf{R} -tree with an isometric action of F_n . In this paper we always assume the action to be very small. We usually assume that the action on T is minimal. But we will also need to consider the metric completion \bar{T} of T , which is minimal if and only if T is simplicial.

In T_0, T and \bar{T} , we write $d(P, Q)$ or $|PQ|$ for the distance between two points (length of the segment PQ). We also write $|e|$ for the length of an edge of T_0 .

We consider F_n -equivariant maps $f : T_0 \rightarrow T$, often requiring that the restriction of f to each edge be isometric, or linear. Note that f is necessarily onto if T is minimal. A segment $PQ \subset T_0$ is *f-backtracking* if $f(P) = f(Q)$.

We say that $f : T_0 \rightarrow T$ has the *bounded backtracking property* (BBT) if there exists a constant $C \geq 0$ such that the f -image of any segment $PQ \subset T_0$ is contained in the C -neighbourhood of the segment $f(P)f(Q) \subset T$. The smallest such C is the *BBT-constant of f* , denoted $\text{BBT}(f)$. Note that $\text{BBT}(f)$ does not depend on the metric on T_0 . Also note that the image of any f -backtracking segment has diameter at most $2\text{BBT}(f)$.

We note the following fact (see [4, 9, 11]).

Lemma 2.1. *Let T be an \mathbf{R} -tree with a minimal very small action of F_n . Let T_0 be a free simplicial F_n -tree, and $f : T_0 \rightarrow T$ an equivariant map isometric on edges. Then f has bounded backtracking, with $\text{BBT}(f) \leq v(T_0)$.*

In the rest of this section, and in the next one, we consider a very small F_n -tree T such that some (hence every) orbit is dense. Such a tree has trivial arc stabilizers (as will be recalled in Lemma 4.2).

Proposition 2.2. *Let T be a minimal F_n -tree with dense orbits and trivial arc stabilizers. Given $\varepsilon > 0$, there exists a free simplicial F_n -tree T_0 with $v(T_0) < \varepsilon$, and an equivariant map $f : T_0 \rightarrow T$ whose restriction to each edge is isometric.*

Proof. Let $f : T_0 \rightarrow T$ be equivariant and isometric on edges. It suffices to show that, given an edge e of T_0 , we may replace f by $f' : T'_0 \rightarrow T$ with $v(T'_0) \leq v(T_0) - \frac{1}{6}|e|$. By rescaling, we may assume $|e| = 1$.

We first show that *there exists an f -backtracking segment $PQ \subset T_0$ meeting e in a subsegment PR of length $\frac{1}{3}$* (where R is an endpoint of e and P is one of the two points trisecting e). Let M be the midpoint of e . Since orbits in T are dense, there exists a non-trivial $g \in F_n$ such that $d(f(M), f(gM)) \leq \frac{1}{6}$. We let P be the point of e located at distance $\frac{1}{6}$ from M on the side of gM , and Q be a point between P and gM such that

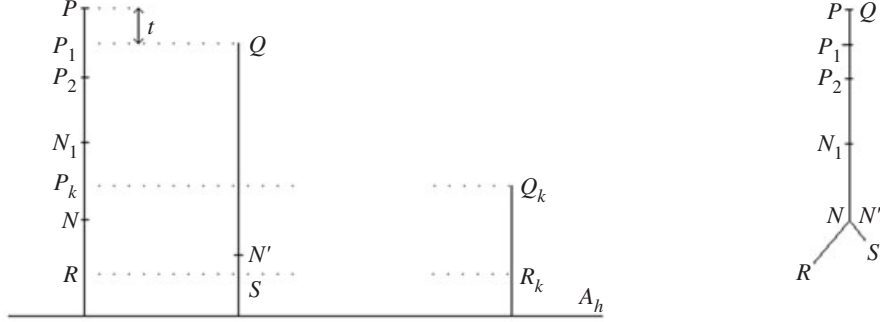


Figure 1.

$f(Q) = f(P)$ (such a Q exists since T is a tree). Note that every interior point x of PR bounds an f -backtracking segment xy with $y \in RQ$.

Consider all segments PQ as above. Since T_0 is a locally finite simplicial tree, we can choose one of minimal length. The easy case is when the interior of RQ does not meet any he , with $h \in F_n$ non-trivial. For then we can redefine f on the orbit of RP and equivariantly fold RP over RQ , obtaining $f' : T'_0 \rightarrow T$ with $v(T'_0) = v(T_0) - \frac{1}{3}$.

Suppose that RQ entirely contains some he . Since every interior point of he bounds an f -backtracking subsegment of PQ , we can replace PQ by a shorter f -backtracking segment, contradicting the choice of PQ .

We therefore reduce to the following situation: RQ intersects some he in a subsegment SQ , and the interior of RS is disjoint from the orbit of e . Minimality of PQ implies $|SQ| \leq \frac{1}{3}$ (otherwise there would be a shorter backtracking segment starting at the point of SQ at distance $\frac{1}{3}$ from S). Orient e and he by choosing an orientation of PQ .

First suppose that h maps e to he in an orientation-preserving way (i.e. $hR \neq S$). Then $h(RP)$ does not meet RQ and we can fold. Now suppose that h reverses orientation. Then the interior of PR is disjoint from the translation axis A_h of h , and for every integer $k \geq 1$ the intersection of the segment $Ph^k(P)$ with the orbit of e is the union of the two segments PR and $h^k(PR)$.

Note that no interior point Q' of PQ is mapped by f onto $f(P) = f(Q)$. This implies that $f(PR)$ and $f(QS)$ intersect along a non-degenerate arc $f(PN) = f(QN')$.

If $|NR| \geq \frac{1}{6}$ (in particular, if $|SQ| \leq \frac{1}{6}$), we choose a point $N'' \in RS$ with $f(N'') = f(N)$ and we fold RN over RN'' , obtaining T'_0 with $v(T'_0) = v(T_0) - |NR| \leq v(T_0) - \frac{1}{6}$.

Assume therefore $|NR| < \frac{1}{6} < |SQ|$ (see figure 1, showing points in T_0 on the left, and their images by f on the right). Let $t = \frac{1}{3} - |SQ|$ (recall that t is non-negative). Let $P_1 = h^{-1}(Q)$ be the point of PR at distance t from P . Then h maps RP_1 onto SQ in T_0 , and $f(RP_1)$ onto $f(SQ)$ in T . In particular, h acts as a translation by t on $f(PN)$. This implies $t > 0$, since otherwise h would fix the non-degenerate segment $f(PN)$. Let $k \geq 2$ be the smallest integer such that $kt \geq \frac{1}{6}$.

For $1 \leq i \leq k$, let P_i be the point of PR at distance it from P . We have $f(hP_i) = f(P_{i-1})$ and therefore $f(h^i P_i) = f(Q)$. We let $Q_k = h^k P_k$ and $R_k = h^k R$.

Then $f(Q_k) = f(h^k P_k) = f(Q)$, and so PQ_k is an f -backtracking segment. As pointed out earlier, the intersection of PQ_k with the orbit of e consists of PR and $Q_k R_k$. Now $|Q_k R_k| = |P_k R| = \frac{1}{3} - |PP_k| \leq \frac{1}{6}$, and therefore $|f(Q)f(R_k)| \leq \frac{1}{6}$. If N_1 denotes the mid-point of PR , then $f(N_1)$ separates $f(R)$ from $f(R_k)$. We conclude as above, choosing $N'' \in RR_k$ with $f(N'') = f(N_1)$ and folding RN_1 over RN'' . \square

Lemma 2.1 and Proposition 2.2 immediately imply the following result.

Corollary 2.3. *Let T be a minimal F_n -tree with dense orbits and trivial arc stabilizers. Given $\varepsilon > 0$, there exists a free simplicial F_n -tree T_0 and an equivariant map $f : T_0 \rightarrow T$ with $\text{BBT}(f) < \varepsilon$.*

Remark 2.4. This corollary may be extended to arbitrary actions with trivial arc stabilizers, but not to arbitrary very small actions (see Remark 3.3).

Corollary 2.5. *Let T be as above. Given $P \in T$ and $\varepsilon > 0$, there exists a basis $\{a_1, \dots, a_n\}$ of F_n such that $\sum_{i=1}^n d(P, a_i P) < \varepsilon$.*

Proof. Take $f : T_0 \rightarrow T$, with $v(T_0) < \varepsilon/2n$. Fix $P_0 \in T_0$ with $f(P_0) = P$, and choose a basis of $\pi_1(T_0/F_n, \pi(P_0))$ represented by loops of length less than $2v(T_0)$. \square

This corollary readily extends to points P in the completion \bar{T} . Combined with Lemma 2.1, it implies the following.

Remark 2.6. Given $P \in \bar{T}$ and $\varepsilon > 0$, there exists a Cayley tree Γ of F_n and $f : \Gamma \rightarrow \bar{T}$ with $\text{BBT}(f) < \varepsilon$ sending the vertex g to gP .

3. The point $Q(X)$

Given an R -tree S , we define ∂S as the set of equivalence classes of rays $\rho : [0, +\infty) \rightarrow S$, where ρ is an isometric map and two rays are equivalent if their images have infinite intersection. If S is a simplicial tree with a free F_n -action, there is a canonical identification of ∂S with ∂F_n .

Now fix T (as in Proposition 2.2) and $X \in \partial F_n$. We always denote by $f : T_0 \rightarrow T$ an equivariant map with T_0 free simplicial. We represent X by a ray ρ in T_0 and we consider $r = f \circ \rho$. We will usually confuse a ray ρ and its image, and similarly r and its image, thus writing $r = f(\rho)$.

We say that X is T -bounded if r is bounded in T (this does not depend on the choice of f , as follows from [11, §3] or from the next proof).

If r is not bounded, then clearly it lies in the $\text{BBT}(f)$ -neighbourhood of a ray ρ' . This situation was studied in [11], with the notation $X = j(\rho')$. Here we shall go in the other direction, defining $Q(X)$ as the point of ∂T corresponding to ρ' .

Now we consider the case when r is bounded.

Proposition 3.1. *Let T be a minimal F_n -tree with dense orbits and trivial arc stabilizers. Suppose $X \in \partial F_n$ is T -bounded. Then there exists a unique point $Q(X) \in \bar{T}$ such that, for any $f : T_0 \rightarrow T$ and any ray ρ representing X in T_0 , the point $Q(X)$ belongs*

to the closure of $f(\rho)$ in \bar{T} . Furthermore, every $f(\rho)$ is contained in the $2\text{BBT}(f)$ -ball centred at $Q(X)$, except for an initial part.

Recall that \bar{T} denotes the metric completion of T . We say that $f(\rho)$ is contained in a set A except for an initial part if $f(\rho(t)) \in A$ for all t larger than some t_0 .

Proof. (i) First consider two maps $f : T_0 \rightarrow T$ and $f' : T'_0 \rightarrow T$. We may assume that they are isometric on edges. Let C be the backtracking constant of f . Subdivide T'_0 equivariantly so that all edges now have length less than C . Given a vertex v of the subdivided T'_0 , let $\zeta(v)$ be a point of T_0 such that $f(\zeta(v)) = f'(v)$. We may choose $\zeta(v)$ in an equivariant way, and extend ζ to an equivariant map $\zeta : T'_0 \rightarrow T_0$ which is linear on each edge. Given an edge vw of T'_0 , its image by $f \circ \zeta$ is contained in the C -neighbourhood of the segment $f'(v)f'(w)$. This implies that $f \circ \zeta$ is $2C$ -close to f' (recall that $|f'(v)f'(w)| \leq C$).

Represent $X \in \partial F_n$ by rays ρ, ρ' in T_0, T'_0 . Since $\zeta(\rho')$ contains ρ (after truncating if needed), we find that, *except for an initial part*, $r = f(\rho)$ is contained in the $2\text{BBT}(f)$ -neighbourhood of $r' = f'(\rho')$. Similarly, r' is contained in the $2\text{BBT}(f')$ -neighbourhood of r . In particular, boundedness of r depends only on X .

(ii) Now fix f and let $C + \delta > C = \text{BBT}(f)$. We show that *except for an initial part*, r is entirely contained in a ball of radius $C + \delta$. Since X is assumed to be T -bounded, we may consider $\eta = \sup_{t \geq 0} d(r(0), r(t))$. Choose t_0 with $d(r(0), r(t_0)) > \eta - \delta$. If $C \geq \eta - \delta$, the whole of r is contained in the $(C + \delta)$ -ball centred at $r(0)$. If not, let x be the point of the segment $r(0)r(t_0)$ at distance C from $r(t_0)$. For $t \geq t_0$ the point x separates $r(0)$ from $r(t)$. Since $d(r(0), x) > \eta - \delta - C$ and $d(r(0), r(t)) \leq \eta$ we obtain $d(x, r(t)) \leq C + \delta$.

(iii) It is now easy to conclude. Choose a sequence ε_n converging to 0, as well as maps $f_n : T_n \rightarrow T$ with $\text{BBT}(f_n) < \varepsilon_n$ (given by Corollary 2.3). Represent X by rays ρ_n in T_n such that $r_n = f_n(\rho_n)$ is contained in an ε_n -ball. As n increases, the distance between these balls goes to 0 and therefore they converge to a unique $Q(X) \in \bar{T}$. For every $r = f(\rho)$ as above, the point $Q(X)$ belongs to the closure of r since r_n is contained in the $2\varepsilon_n$ -neighbourhood of r .

To prove the ‘furthermore’, choose t_0 with $d(Q(X), f(\rho(t_0))) < \text{BBT}(f)$ and suppose there exists $t_1 > t_0$ with $d(Q(X), f(\rho(t_1))) > 2\text{BBT}(f)$. For $t \geq t_1$ the point located at distance $\text{BBT}(f)$ of $f(\rho(t_1))$ on the segment $Q(X)f(\rho(t_1))$ separates $Q(X)$ from $f(\rho(t))$, a contradiction. \square

We can now associate a point $Q(X)$ to every $X \in \partial F_n$. If X is T -bounded, it is the point of \bar{T} provided by Proposition 3.1. If not, then $Q(X)$ is the point of ∂T defined by $r = f(\rho)$, as explained before the statement of Proposition 3.1. It is the unique point of ∂T such that $X = j(Q(X))$ (see [11, Lemma 3.5]).

The map $Q : \partial F_n \rightarrow \bar{T} \sqcup \partial T$ thus defined is obviously F_n -equivariant. The map Q restricts to a bijection from the set of T -unbounded points of ∂F_n onto ∂T .

Remark 3.2. Suppose T is dual to an arational measured geodesic lamination on a compact hyperbolic surface Σ with geodesic boundary. There is a $\pi_1 \Sigma$ -equivariant quotient map from the universal covering $\tilde{\Sigma}$ onto T . An infinite geodesic ray in $\tilde{\Sigma}$ maps onto an

open segment in T . The corresponding $Q(X)$ is the endpoint of that segment (in ∂T if it has infinite length, in \bar{T} otherwise).

Remark 3.3. Suppose T is a very small simplicial F_n -tree (more generally, a simplicial G -tree where G is a word hyperbolic group and edge stabilizers are quasiconvex (see [5])). Then ∂F_n is naturally identified to the disjoint union of ∂T and the union of the boundaries of vertex stabilizers [5, Proposition 1.3]. If edge stabilizers are trivial, these boundaries are disjoint and one may define a map Q as above. If $g \in F_n$ fixes an edge, though, one cannot associate a single point of T to $X = \lim_{p \rightarrow +\infty} g^p$. This is due to the failure of Corollary 2.3 in this case.

Lemma 3.4. *Let $X \in \partial F_n$ be T -bounded. For any $P \in \bar{T}$, there exists a sequence $g_p \in F_n$ such that $g_p \rightarrow X$ and $g_p P \rightarrow Q(X)$. Conversely, if $h_p \rightarrow X$ and $h_p P$ converges to some point $R \in \bar{T}$, then $R = Q(X)$.*

Proof. Given $\varepsilon > 0$, choose $f : \Gamma \rightarrow \bar{T}$ as in Remark 2.6. Long initial subwords of X (in the basis corresponding to Γ) provide elements $g \in F_n$ arbitrarily close to X with $d(gP, Q(X)) < 2\varepsilon$. Conversely, assume $R \neq Q(X)$ and choose f as in Remark 2.6 with $\text{BBT}(f)$ small with respect to $d(R, Q(X))$. Let $h \in F_n$ be the maximal initial subword common to h_p and h_q , for p and q large. The point hP is $2\text{BBT}(f)$ -close to both R and $Q(X)$, a contradiction. \square

It is not true, however, that $h_p P \rightarrow Q(X)$ for *every* sequence $h_p \rightarrow X$.

Corollary 3.5. *The equivariant map $Q : \partial F_n \rightarrow \bar{T} \sqcup \partial T$ is onto.*

Proof. We have to show that every $R \in \bar{T}$ is a $Q(X)$. Choose any $P \in T$ and any sequence h_p such that $h_p P \rightarrow R$. Any limit point X of h_p satisfies $Q(X) = R$. \square

Remark 3.6. We do not know whether Q is finite-to-one when the action on T is free.

We now consider two distinct points $X, X' \in \partial F_n$. Let $f : T_0 \rightarrow T$ be as above. If X, X' are T -bounded and γ is the bi-infinite geodesic joining X to X' in T_0 , then $f(\gamma)$ is contained in the $\text{BBT}(f)$ -neighbourhood of the segment joining $Q(X)$ to $Q(X')$.

Remark 3.7. In particular, if $Q(X) = Q(X')$, then $f(\gamma)$ is contained in the $\text{BBT}(f)$ -ball centred at $Q(X)$.

Given $X, X' \in \partial F_n$, we define $d_T(Q(X), Q(X'))$ as 0 if $X = X'$, as the distance between $Q(X)$ and $Q(X')$ in \bar{T} if both X and X' are T -bounded, and as $+\infty$ in the remaining cases. This gives us a map κ , with values in $[0, +\infty]$, defined on the product of $(\partial F_n)^2$ with the space of F_n -trees with dense orbits and trivial arc stabilizers.

Proposition 3.8. *The map κ is lower semicontinuous.*

Proof. Given $C < d_T(Q(X), Q(X'))$, we have to show that κ remains bigger than C under perturbation. Choose ε small with respect to $d_T(Q(X), Q(X')) - C$. Fix $f : T_0 \rightarrow T$ isometric on edges, with $v(T_0) < \varepsilon$. For T' close to T , there exists $f' : T_0 \rightarrow T'$, linear on

edges, such that images of edges have approximately the same length in T and T' . The map f' has Lipschitz constant close to 1 and BBT less than $2v(T_0)$.

Let $\gamma \subset T_0$ be the bi-infinite geodesic joining the ends X and X' . Choose points A, B on γ with $d(f(A), f(B))$ close to $d_T(Q(X), Q(X'))$ (closeness is measured with respect to ε). If a bi-infinite geodesic joining two ends Y, Y' in T_0 contains the segment between A and B , then for T' close enough to T its image by f' has diameter bigger than C , and therefore $d_{T'}(Q(Y), Q(Y')) > C$. \square

4. Train tracks, laminations, trees

Let α be an IWIP automorphism of F_n . Let $\Phi : \tau \rightarrow \tau$ be a train track map representing α (see [2]). We denote by $\tilde{\tau}$ the universal covering of the graph τ . It is equipped with a free action of F_n , and a lift φ of Φ such that $\alpha(g) \circ \varphi = \varphi \circ g$ for $g \in F_n$.

We define the *expanding lamination* Λ as an F_n -invariant collection of geodesic segments in $\tilde{\tau}$, called *leaf segments*, as follows (see [4, 13]). First, a compact segment γ is in Λ if and only if it is contained in $\varphi^k(e)$ for some $k \geq 1$ and some edge e . If $\gamma \in \Lambda$, then there exists k_0 such that $\varphi^k(e)$ contains a translate of γ for every edge e and every $k \geq k_0$ (this follows from irreducibility of α and its powers). Next, we say that a half-infinite or bi-infinite geodesic is in Λ if every compact subsegment is in Λ . A bi-infinite $\gamma \in \Lambda$ is called a *leaf* of Λ .

We note the following simple facts.

Lemma 4.1.

- (1) If $\gamma \in \Lambda$, then $\varphi|_\gamma$ is injective and $\varphi(\gamma) \in \Lambda$.
- (2) Every $\gamma \in \Lambda$ is contained in a bi-infinite $\gamma' \in \Lambda$.
- (3) If a bi-infinite γ is in Λ , then the bi-infinite geodesic θ such that $\gamma = [\varphi(\theta)]$ (tightened image) is in Λ ; in particular, $\varphi|_\theta$ is injective: $\varphi(\theta) = \gamma$.
- (4) If e, f are edges with a common initial vertex v , there exists a sequence $e = e_0, e_1, \dots, e_p = f$ of distinct edges starting at v such that all edge paths $\bar{e}_i e_{i+1}$ are in Λ (connectedness of the Whitehead graph, see [4, 13]).

Since the set of ends of $\tilde{\tau}$ is canonically identified with ∂F_n , the set of leaves of Λ may be viewed in a more intrinsic way, as an F_n -invariant collection of unordered pairs $\{X, X'\}$ of distinct elements of ∂F_n . This collection (still denoted Λ) depends only on α (not on τ and Φ), see [4].

We define the *support* $s(\Lambda) \subset \partial F_n$ as the set of $X \in \partial F_n$ such that Λ contains some pair $\{X, X'\}$. It is F_n -invariant and α -invariant.

We also consider the *contracting lamination* of α (the expanding lamination of α^{-1}). We usually write Λ^+ for the expanding lamination and Λ^- for the contracting one.

Outer space may be viewed as the space of projective classes of free simplicial actions of F_n on \mathbf{R} -trees (see [22]). Compactified outer space $\overline{CV_n}$ is the space of projective classes of very small actions. We write $[T]$ for the point of $\overline{CV_n}$ determined by a very small \mathbf{R} -tree T .

Lemma 4.2. *If a very small F_n -tree T has dense orbits, then all arc stabilizers are trivial.*

Proof. This is folklore (compare [19, Proposition 1.4], [1, proof of Theorem 2.2]). A proof appears in [10, Proposition I.10] when T is geometric. We sketch a proof for T not geometric. We first show (with the notations of [10]) that for every $x \in T$ the action of $\text{Stab } x$ on $\pi_0(T \setminus \{x\})$ has finitely many orbits. Otherwise, for any N , we may approximate T by a small, minimal, geometric tree T' with a branch point y having at least N $\text{Stab } y$ -orbits of directions with cyclic stabilizer (see [10]). These directions are contained in the simplicial part of T' by [10, Proposition I.10]. But there is a uniform bound for the valence of a vertex in a small minimal graph of groups decomposition of F_n , and therefore N is bounded.

Now let a non-trivial $g \in F_n$ fix a non-degenerate segment $I \subset T$. Branch points of T are dense in I . By the above and [10, Corollary III.3], there exists $h \in F_n$ carrying a subsegment $J \subset I$ onto another subsegment of I , in an orientation-preserving way. Then g and $h^{-1}gh$ generate a free group of rank 2 fixing J , a contradiction. \square

The automorphism α acts on $\overline{CV_n}$ by precomposing the action with α . Its fixed points $[T^+]$ and $[T^-]$ may be constructed as follows (see [11] for a detailed exposition).

Let $\Phi, \tilde{\tau}, \varphi$ be as above. The transition matrix of Φ is irreducible, with a Perron–Frobenius eigenvalue $\lambda > 1$. Using an eigenvector associated to λ , one may assign to each segment β of $\tilde{\tau}$ a *Perron–Frobenius length* $\|\beta\|$, in such a way that $\|\varphi(\beta)\| = \lambda\|\beta\|$ if β is a leaf segment of Λ^+ (and $\|\varphi(\beta)\| \leq \lambda\|\beta\|$ for arbitrary β).

The \mathbf{R} -tree T^+ may now be defined as the metric space associated to the pseudo-distance d_+ on $\tilde{\tau}$ giving length $d_+(\beta) = \lim_{p \rightarrow +\infty} \|\varphi^p(\beta)\|/\lambda^p$ to a segment β . One constructs T^- similarly, using a train track map associated to α^{-1} .

The length function $\ell_+ : F_n \rightarrow \mathbf{R}$ of the action of F_n on T^+ satisfies $\ell_+ \circ \alpha = \lambda\ell_+$. If T is a very small F_n -tree, with length function ℓ_T , saying that $\alpha^p([T])$ converges to $[T^+]$ means that there exists a sequence c_p such that $(1/c_p)\ell_T \circ \alpha^p$ converges to ℓ_+ .

5. The main argument

Let $\alpha \in \text{Aut } F_n$ be irreducible with irreducible powers. Let Λ^+, Λ^- be the expanding and contracting lamination. The goal of this section is the following statement.

Proposition 5.1. *Let T be a minimal F_n -tree with dense orbits and trivial arc stabilizers. There exists a leaf $\{X, X'\}$ of Λ^+ or of Λ^- such that $Q(X) \neq Q(X')$.*

Recall that $Q(X) = Q(X')$ is possible only if X and X' are T -bounded (because the map $j : \partial T \rightarrow \partial F_n$ constructed in [11] satisfies $j \circ Q = \text{id}$).

Let $\tilde{\tau}$ be as in § 4. We view it as a metric tree, with each edge having length 1 (we do not use Perron–Frobenius length in this section). The proof of Proposition 5.1 relies on two lemmas.

Lemma 5.2. *Suppose $Q(X) = Q(X')$ for every leaf $\{X, X'\}$ of Λ^+ . Let $Y, Y' \in \partial F_n$ belong to the support $s(\Lambda^+)$. Then the distance in \bar{T} between $Q(\alpha^p(Y))$ and $Q(\alpha^p(Y'))$ tends to 0 as $p \rightarrow +\infty$.*

Proof. Let $\gamma \subset \tilde{\tau}$ be the bi-infinite geodesic with ends Y, Y' . The hypothesis that $Y, Y' \in s(\Lambda^+)$ means that the complement of some compact subsegment consists of two half-infinite rays ρ, ρ' contained in Λ^+ .

Let e, e' be the first edges of ρ and ρ' . Using the last assertion of Lemma 4.1, we can find a finite sequence $\gamma_1, \dots, \gamma_m$ of edge paths $\gamma_i = e_i e'_i$ of length 2, belonging to Λ^+ , with $e_1 = \bar{e}$ and $e'_m = e'$, such that for each i the edges e'_i and e_{i+1} are the same, but not necessarily with the same orientation (thus the union of γ_i and γ_{i+1} is either a tripod or a segment of length 3).

Given ε , choose $f : T_0 \rightarrow T$ as in Corollary 2.3, with $\text{BBT}(f) < \varepsilon$. Next choose an equivariant map $\mu : \tilde{\tau} \rightarrow T_0$ and note that there exists a constant C such that, if two bi-infinite geodesics in $\tilde{\tau}$ have intersection of length greater than C , then their tightened images in T_0 have non-empty intersection.

For p large, the image of each edge of $\tilde{\tau}$ by φ^p has length greater than C . We complete the proof by showing that the distance between $Q(\alpha^p(Y))$ and $Q(\alpha^p(Y'))$ is at most $2(m+2)\varepsilon$. Let $\lambda_0, \lambda_1, \dots, \lambda_m, \lambda_{m+1} \in \Lambda^+$ be bi-infinite geodesics containing $\rho, \gamma_1, \dots, \gamma_m, \rho'$, respectively. Let $\delta_i \subset T_0$ be the tightened image of $\varphi^p(\lambda_i)$ by μ .

Since λ_i and λ_{i+1} have an edge in common, our choice of p guarantees that δ_i and δ_{i+1} have non-empty intersection. Furthermore, the ends X_i, X'_i of δ_i satisfy $Q(X_i) = Q(X'_i)$, and so $f(\delta_i) \subset T$ is contained in the ε -ball centred at $Q(X_i)$ (see Remark 3.7). We conclude that $d(Q(Z), Q(Z')) \leq 2(m+2)\varepsilon$, where Z, Z' are the ends determined by $\varphi^p(\rho)$ and $\varphi^p(\rho')$. But the relation $\alpha(g) \circ \varphi = \varphi \circ g$ implies $Z = \alpha^p(Y)$ and $Z' = \alpha^p(Y')$. \square

Lemma 5.3. *Suppose $Q(X) = Q(X')$ for every leaf $\{X, X'\}$ of Λ^- . There exist maps $i_p : \tilde{\tau} \rightarrow \bar{T}$ ($p \in \mathbf{N}$) such that $i_p \circ \varphi^p$ is F_n -equivariant and $\text{BBT}(i_p) \rightarrow 0$ as $p \rightarrow +\infty$.*

Proof. Fix X in the support of Λ^- . Also fix an equivariant map π from $\tilde{\tau}$ to a Cayley tree Γ of F_n , obtained by collapsing edges in the lift of a maximal subtree of τ . Choose a base vertex $v \in \Gamma$. Define i_0 as $j_0 \circ \pi$, where $j_0 : \Gamma \rightarrow \bar{T}$ is equivariant, linear on edges, and maps v to $Q(X)$. We define i_p analogously as $j_p \circ \pi$, but now j_p is required to map v to $Q(\alpha^{-p}(X))$ and to satisfy the twisted equivariance property $g \circ j_p = j_p \circ \alpha^p(g)$ (which expresses that $i_p \circ \varphi^p$ is equivariant).

We check that $\text{BBT}(i_p) \rightarrow 0$. By Lemma 2.1, it suffices to show that the length of $j_p(e)$ tends to 0 for every edge e of Γ . We may assume that e has vertices v and gv , with $g \in F_n$. Now $j_p(v) = Q(\alpha^{-p}(X))$ and

$$j_p(gv) = \alpha^{-p}(g)j_p(v) = \alpha^{-p}(g)Q(\alpha^{-p}(X)) = Q(\alpha^{-p}(gX)).$$

Since X and gX both belong to the support of Λ^- , we conclude by applying Lemma 5.2 to α^{-1} . \square

Proof of Proposition 5.1. We argue by way of contradiction, assuming that $Q(X) = Q(X')$ for every leaf $\{X, X'\}$ of Λ^+ or Λ^- . Let i_p be given by Lemma 5.3. Let e be an

edge of $\tilde{\tau}$. If $\gamma \in \Lambda^+$ is a bi-infinite geodesic containing e , then $\varphi^p(\gamma)$ is also a leaf of Λ^+ and therefore its image by i_p is contained in a ball of radius $\text{BBT}(i_p)$. It follows that the diameter of $(i_p \circ \varphi^p)(e)$ is bounded by $2\text{BBT}(i_p)$. If u is a conjugacy class in F_n , represented by a loop of length k in τ , its translation length in T is bounded by $2k\text{BBT}(i_p)$ for all p (because $i_p \circ \varphi^p$ is equivariant). Thus every u has translation length 0 in T , a contradiction. \square

6. Conclusion

In this section we deduce the main theorem from Proposition 5.1 and the following result (proved in the next section).

Proposition 6.1 (cf. Lemmas 3.4 and 3.5.1 of [4]). *Let T be an \mathbf{R} -tree with a minimal very small action of F_n . Suppose there exist a free simplicial F_n -tree T_0 , an equivariant map $f : T_0 \rightarrow T$, and a bi-infinite geodesic $\gamma_0 \subset T_0$ representing a leaf of Λ^+ such that $f(\gamma_0)$ has diameter greater than $2\text{BBT}(f)$. Then $f(\gamma_0)$ has infinite diameter and there exists a neighbourhood V of $[T]$ in $\overline{CV_n}$ such that $\alpha_{|V}^p$ converges to $[T^+]$ uniformly as $p \rightarrow +\infty$.*

We first show that every very small T satisfies the hypothesis of Proposition 6.1, provided we allow γ_0 to be a leaf of Λ^- or Λ^+ . We distinguish several cases.

If T has dense orbits, it has trivial arc stabilizers by Lemma 4.2. Proposition 5.1 provides $\{X, X'\} \in \Lambda^\pm$ with $Q(X) \neq Q(X')$. We choose $f : T_0 \rightarrow T$ with $2\text{BBT}(f) < d(Q(X), Q(X'))$, using Corollary 2.3, and we let γ_0 be the geodesic joining the ends of T_0 corresponding to X and X' .

Next suppose that orbits are not dense, but T is not simplicial. Then T contains simplicial pieces, as well as non-degenerate subtrees T_v with the property that some subgroup $G_v \subset F_n$ acts on T_v with dense orbits (see [12]). In particular, there exists an equivariant collapsing map $\pi : T \rightarrow T_m$ where T_m is a very small F_n -tree with dense orbits. Choose X, X' as above, using T_m . By Proposition 6.1, at least one of X, X' (both, in fact) is not T_m -bounded. Thus it is not T -bounded and $f : T_0 \rightarrow T$ may be chosen arbitrarily.

The last case is when T is simplicial. It suffices to show that X is T -unbounded whenever $\{X, X'\} \in \Lambda^+$. As pointed out in Remark 3.3, ∂F_n is the disjoint union of ∂T and the boundaries of the vertex stabilizers. A vertex stabilizer H is finitely generated and has infinite index. The proof of Proposition 2.4 in [4] shows that X cannot belong to ∂H (consider a half-leaf rather than a whole leaf). Thus X is T -unbounded.

Given any T as in Proposition 6.1, we now know that $\lim_{p \rightarrow \infty} \alpha^p([T]) = [T^+]$ or $\lim_{p \rightarrow \infty} \alpha^{-p}([T]) = [T^-]$. We also know that $[T^+]$ is an attracting fixed point, because α^p converges to $[T^+]$ uniformly on a neighbourhood of $[T^+]$ as $p \rightarrow +\infty$. Similarly, $[T^-]$ is a repelling fixed point. North-south dynamics follow easily.

Indeed, suppose $[T] \neq [T^-]$. The ω -limit set of $[T]$ (the set of limit points of the sequence $\alpha^p([T])$ as $p \rightarrow +\infty$) is closed and α -invariant. It must therefore contain $[T^+]$ or $[T^-]$. But it cannot contain the repelling point $[T^-]$, so it contains $[T^+]$. It follows that

$\alpha^p([T])$ tends to $[T^+]$. Similarly, $\alpha^{-p}([T])$ tends to $[T^-]$ if $[T] \neq [T^+]$. This completes the proof of the main theorem.

7. Proof of Proposition 6.1

Let $\varphi : \tilde{\tau} \rightarrow \tilde{\tau}$ be as in §4. Fix an equivariant map $\mu : \tilde{\tau} \rightarrow T_0$ and denote by γ the bi-infinite geodesic of $\tilde{\tau}$ such that γ_0 is the tightened image of γ . If $AB \subset \tilde{\tau}$ is a segment, we denote by $d_T(AB)$ the length of the tightened image of AB under $\nu = f \circ \mu$ (i.e. the distance between $\nu(A)$ and $\nu(B)$).

Let A_0, B_0 be points of γ_0 whose images in T have distance bigger than $2\text{BBT}(f)$. Let σ be the central subsegment of $f(A_0)f(B_0)$ of length $d(f(A_0), f(B_0)) - 2\text{BBT}(f)$. If $A'_0B'_0 \subset T_0$ is a segment containing A_0B_0 , the segment $f(A'_0)f(B'_0)$ contains σ .

Since μ has bounded backtracking, we can choose $AB \subset \gamma$ such that $\mu(A')\mu(B')$ contains A_0B_0 for any $A'B' \subset \tilde{\tau}$ containing AB . As a consequence, the tightened image of any such $A'B'$ by ν contains σ . Furthermore, if a segment in $\tilde{\tau}$ contains disjoint subsegments which are translates of AB by elements g_i of F_n , then its tightened image in T contains the $g_i\sigma$ as disjoint subsegments.

Choose m_0 such that $\varphi^{m_0}(e)$ contains a translate of AB for every edge e of $\tilde{\tau}$. If β is any leaf segment contained in Λ^+ , then $d_T(\varphi^{m_0}(\beta)) \geq |\sigma||\beta|$ (with $|\beta|$ the simplicial length of β in $\tilde{\tau}$, and $|\sigma|$ the length of σ in T).

In particular, $f(\gamma_0)$ has infinite diameter, and for every edge e of $\tilde{\tau}$ the length $d_e(p) = d_T(\varphi^p(e))$ tends to infinity with p .

From now on we equip $\tilde{\tau}$ with a Perron–Frobenius length, characterized by the fact that for every edge path $\beta \subset \Lambda^+$ the image $\varphi(\beta)$ has length $\lambda\|\beta\|$ (where λ is the Perron–Frobenius eigenvalue of the transition matrix of the map Φ).

Lemma 7.1. *There exists a number $c > 0$ such that*

$$\lim_{p \rightarrow \infty} \frac{d_T(\varphi^p(\beta))}{\lambda^p \|\beta\|} = c$$

for every edge path $\beta \in \Lambda^+$.

Proof. Let E be the set of edges of a fundamental domain for the action of F_n on $\tilde{\tau}$. For given p , the segment $\varphi^p(\beta)$ is a union of translates of edges of E . Let N_e^p be the number of occurrences of translates of a given $e \in E$. The numbers N_e^p are the components of the image by the p th power of the transition matrix of a fixed non-negative vector (whose components are numbers of occurrences of edges in β). Let $p \rightarrow \infty$. By Perron–Frobenius theory, the sequence N_e^p/λ^p has a positive limit, of the form $c_e k_\beta$ with c_e depending only on e and k_β only on β . Since $\|\varphi^p(\beta)\| = \lambda^p \|\beta\| = \sum_{e \in E} N_e^p \|e\|$, we obtain $k_\beta = \|\beta\|$ (up to a normalization). Thus $N_e^p/\lambda^p \|\beta\| \rightarrow c_e$ as $p \rightarrow \infty$.

Given $\varepsilon > 0$, fix p_0 such that every $d_e(p_0) = d_T(\varphi^{p_0}(e))$ is bigger than $(1/\varepsilon)\text{BBT}(\nu)$. Now $\varphi^{p+p_0}(\beta)$ is a union of translates of $\varphi^{p_0}(e)$, with $\varphi^{p_0}(e)$ appearing N_e^p times. Mapping into T by ν , we obtain

$$\sum_{e \in E} N_e^p (d_e(p_0) - 2\text{BBT}(\nu)) \leq d_T(\varphi^{p+p_0}(\beta)) \leq \sum_{e \in E} N_e^p d_e(p_0).$$

Dividing by $\lambda^{p+p_0}\|\beta\|$, we get, as p goes to infinity,

$$(1 - 2\varepsilon) \sum_{e \in E} \frac{c_e d_e(p_0)}{\lambda^{p_0}} \leq \liminf \frac{d_T(\varphi^p(\beta))}{\lambda^p \|\beta\|} \leq \overline{\lim} \frac{d_T(\varphi^p(\beta))}{\lambda^p \|\beta\|} \leq \sum_{e \in E} \frac{c_e d_e(p_0)}{\lambda^{p_0}}$$

for any $\beta \in \Lambda^+$.

Since $d_e(p_0)/\lambda^{p_0}$ is bounded by $\|e\| \text{Lip}(\nu)$, we see that $d_T(\varphi^p(\beta))/(\lambda^p \|\beta\|)$ has a limit, which is positive and independent of β . \square

We now consider an arbitrary edge path $\beta \subset \tilde{\tau}$. We let $d_+(\beta)$ be the limit of the non-increasing sequence $\|\varphi^p(\beta)\|/\lambda^p$.

Lemma 7.2. $d_T(\varphi^p(\beta))/(\lambda^p d_+(\beta))$ tends to c as $p \rightarrow \infty$.

Proof. Write β as a concatenation of k paths $\beta_i \in \Lambda^+$. Then the tightened image $\varphi^p(\beta)$ is the concatenation of the $\varphi^p(\beta_i)$, possibly with cancellation. The total amount of cancellation is $\lambda^p \|\beta\| - \|\varphi^p(\beta)\| \leq \lambda^p (\|\beta\| - d_+(\beta))$. Furthermore, if $\varphi^p(\beta_i)$ and $\varphi^p(\beta_{i+1})$ overlap along a distance D , then the cancellation between their tightened images in T is bounded by $D \text{Lip}(\nu) + 2 \text{BBT}(\nu)$. From this we obtain

$$\left| d_T(\varphi^p(\beta)) - \sum_i d_T(\varphi^p(\beta_i)) \right| < \lambda^p (\|\beta\| - d_+(\beta)) \text{Lip}(\nu) + k \text{BBT}(\nu).$$

Dividing by $\lambda^p \|\beta\|$ and using Lemma 7.1, we get

$$\liminf_p \left| \frac{d_T(\varphi^p(\beta))}{\lambda^p \|\beta\|} - c \right| \leq \left(1 - \frac{d_+(\beta)}{\|\beta\|} \right) \text{Lip}(\nu).$$

Now apply this inequality to $\varphi^{p_0}(\beta)$,

$$\liminf_p \left| \frac{d_T(\varphi^{p+p_0}(\beta))}{\lambda^p \|\varphi^{p_0}(\beta)\|} - c \right| \leq \left(1 - \frac{d_+(\varphi^{p_0}(\beta))}{\|\varphi^{p_0}(\beta)\|} \right) \text{Lip}(\nu).$$

For p_0 large, the ratio of $\|\varphi^{p_0}(\beta)\|$ to $\lambda^{p_0} d_+(\beta) = d_+(\varphi^{p_0}(\beta))$ is close to 1 and we get the desired result. \square

It is now easy to prove Proposition 6.1. For $g \in F_n$, let β be a fundamental domain for the action of g on its translation axis in $\tilde{\tau}$. Then $d_T(\varphi^p(\beta))$ is the translation length of $\alpha^p(g)$ in T , and $d_+(\beta)$ is the translation length of g in T^+ . We thus get $(\ell_T \circ \alpha^p)/(c\lambda^p) \rightarrow \ell_+$, and therefore $\alpha^p([T]) \rightarrow [T^+]$ in $\overline{CV_n}$.

This convergence is uniform on some neighbourhood of T . Indeed, for T' close to T , we can find $f' : T_0 \rightarrow T'$ with $\text{Lip}(f')$ and $\text{BBT}(f')$ bounded (as in the proof of Proposition 3.8). Thus $\text{Lip}(\nu')$ and $\text{BBT}(\nu')$ are uniformly bounded (with $\nu' = f' \circ \mu$). Since $f(\gamma_0)$ has infinite diameter, we can choose the same points A, B for all T' close enough to T , with a positive lower bound for the length of the segment σ . Since all estimates given above depend only on $|\sigma|$, $\text{Lip}(\nu)$, and $\text{BBT}(\nu)$, we have local uniformity.

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