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Abstract

The theory of isometric group actions on **R**-trees is extended to actions by homeomorphisms with the following non-nesting property: no group element maps an arc properly into itself. A finitely presented group acting freely by homeomorphisms on an **R**-tree is free abelian or splits over a (possibly trivial) cyclic group.

Introduction

There is now a well-established theory for groups acting isometrically on **R**-trees. This Rips theory, combined with Bass–Serre theory for groups acting on simplicial trees, has found many applications, for instance to hyperbolic groups (see the recent survey [**14**]). A famous example is the finiteness of the group $\mathrm{Out}(G)$ when G is a hyperbolic group that does not split over a virtually cyclic subgroup.

Other applications require studying non-isometric group actions on **R**-trees, namely actions by arbitrary homeomorphisms, or affine actions (every $g \in G$ multiplies the metric by some $\lambda_g \in \mathbf{R}^*$).

For instance, fixed point theorems in spaces such as Culler–Vogtmann's outer space yield affine actions of semi-direct products (see [**15**]). Transversely affine foliations lead to affine actions on **R**-trees obtained as leaf spaces (see [**11**]).

Furthermore, Bowditch has shown in [**2**] that a one-ended hyperbolic group G whose boundary has a global cut point admits a natural action by homeomorphisms on an **R**-tree, and one wants to deduce that G splits over a virtually cyclic subgroup. This was our main motivation, as suggested by a question of Bestvina. A slight extension of Theorem 1 below was used by G. A. Swarup [**19**] in his proof of the 'cut point conjecture' (see Corollary 6). Our paper may be compared with [**3**], which was prepared at about the same time and obtains the splitting through actions on 'monotone trees'.

Here we bring into the theory a famous theorem about foliations proved by Sacksteder in 1965. We show that a non-isometric action of a group G on an **R**-tree gives as much information about G as an isometric one, provided it satisfies the following non-nesting property introduced by Bestvina: *no group element maps an arc properly into itself* ($gI \subseteq I$ for I a non-degenerate segment implies $gI = I$).

Theorem 1. *If a finitely presented group G admits a non-trivial non-nesting action by homeomorphisms on an* **R**-*tree T, then it admits a non-trivial isometric action on some* **R**-*tree T_0. A subgroup fixing an arc in T_0 fixes an arc in T.*

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An action is said to be *non-trivial* if no point of T is fixed by the whole group. Recall [**1**, **14**] that a finitely presented group G with a non-trivial, stable, isometric action on an **R**-tree T splits over an extension of an abelian group by a subgroup fixing an arc of T. We then obtain, for instance, the following.

COROLLARY. *If a finitely presented group G acts freely by homeomorphisms on an* **R**-*tree*, *then G is free abelian or G splits over a* (*possibly trivial*) *cyclic group*.

Note that a free action is always non-nesting. Liousse [**11**] has constructed free affine actions of groups that do not admit free isometric actions (for instance, the group with presentation $\langle a_1, a_2, a_3, b_1, b_2, b_3; [a_1, b_1] = [a_2, b_2] = [a_3, b_3]\rangle$). There is no known classification of finitely presented groups that can act freely by homeomorphisms on **R**-trees.

COROLLARY. *If a hyperbolic group G admits a non-trivial non-nesting action with virtually cyclic arc stabilizers*, *then G splits over a virtually cyclic group*.

Let us now explain Sacksteder's theorem, and sketch how it is used to prove Theorem 1.

Sacksteder's main results may be summarized as follows (see [**16**, **8**]). Let $\mathscr{F}$ be a codimension one foliation of class C^2 on a closed manifold M. Then *every exceptional minimal set contains a leaf with non-trivial linear holonomy; if* $\mathscr{F}$ *has no holonomy*, *then* $\mathscr{F}$ *is topologically conjugate to a foliation defined by a closed differential one-form*.

These results are proved by analysing the holonomy pseudogroup of $\mathscr{F}$. Recall that a *pseudogroup* on a space X is a collection Γ of homeomorphisms $\gamma: U \to V$ between open subsets of X, which is closed under certain operations (composition, inversion, restriction, extension).

We shall use the following statement.

THEOREM 2 [**16**, Theorem 4]. *Let* Γ *be a pseudogroup of homeomorphisms of the circle* S^1. *If every orbit is dense and no* $\gamma \in \Gamma$ *sends an interval properly into itself*, *then there exists a* Γ-*invariant probability measure*.

In order to be self-contained, we shall rephrase this theorem without mentioning pseudogroups (see Theorem 5).

Reducing Theorem 1 to Theorem 2 requires a few steps. First, using a construction due to Rips, we obtain a finite system $\mathscr{K} = \{\phi_i : A_i \to B_i\}$ of homeomorphisms between closed subtrees of a compact tree K with finitely many vertices. We prove that it suffices to construct a non-trivial non-atomic $\mathscr{K}$-invariant measure on K. Passing from $\mathscr{K}$ to a pseudogroup as in Theorem 2 involves collapsing each component of the complement of an infinite minimal $\mathscr{K}$-invariant set. A technical difficulty arises from the fact that domains of elements of $\mathscr{K}$ are compact, while domains of elements of a pseudogroup are required to be open. Though we do not use this language, readers familiar with foliations will easily recognize equivalence of pseudogroups (introduced by Haefliger in [**9**]), and the pseudogroup induced on a closed curve transverse to a foliation.

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1. *Preliminaries: non-nesting actions on trees, systems of maps*

Non-nesting actions on **R**-*trees*. An **R**-tree is a metric space T such that, given $x \neq y$ in T, there exists a unique arc with endpoints x and y, and this arc is isometric to a subinterval of **R** (recall that an *arc* is a space homeomorphic to $[0, 1]$). We let $[x, y]$ be the arc between x and y if $x \neq y$, or the point $\{x\}$ if $x = y$. An **R**-tree may also be defined as a connected 0-hyperbolic metric space.

Remark. Since we shall be studying actions by homeomorphisms, the relevant structure is the underlying topological structure. It may be characterized as being metrizable, uniquely arc-connected, and locally arc-connected [**12**].

Let G be a group acting by homeomorphisms on an **R**-tree T. We shall identify an element $g \in G$ and the corresponding homeomorphism of T. The element $g \in G$ is *elliptic* if the fixed point set $\operatorname{Fix} g$ is non-empty, *hyperbolic* otherwise.

The action is *trivial* if some $x \in T$ is fixed by the whole group. It is *non-nesting* if no $g \in G$ maps an arc properly into itself. This is equivalent to saying that every elliptic g acts freely on $\pi_0(T \backslash \operatorname{Fix} g)$. Every isometric action, every free action, is non-nesting.

Non-nesting actions have the same basic properties as isometric actions, which are summarized in the following statement.

Theorem 3. *Let T be an* **R**-*tree with a non-nesting action of a group G.*

(1) *Let $g \in G$ be elliptic. Then* $\operatorname{Fix} g \subset T$ *is a closed subtree. If* $x \notin \operatorname{Fix} g$, *then* $\operatorname{Fix} g \cap [x, gx]$ *consists of precisely one point.*

(2) *If g is hyperbolic, then it has an axis: there is a proper embedding $i : \mathbf{R} \to T$ such that $A = i(\mathbf{R})$ is g-invariant and the action of g on A is topologically conjugate to a translation.*

(3) *If G is finitely generated and the action is non-trivial, then there are hyperbolic elements. The union $T_{\min}$ of the axes of the hyperbolic elements is the smallest non-empty G-invariant subtree.*

Proof. This theorem is well-known for isometric actions (see [**13**, **4**, **17**, **18**]). We sketch the arguments for completeness, insisting on the differences with the isometric case.

The first part of assertion (1) is clear: if g fixes two points x, y, then it fixes the whole arc $[x, y]$. Given $x \notin \operatorname{Fix} g$, choose $x_0 \in \operatorname{Fix} g$, and define u by $[x_0, x] \cap \operatorname{Fix} g = [x_0, u]$. Non-nesting implies $\operatorname{Fix} g \cap [x, gx] = \{u\}$.

Now suppose that g is hyperbolic. Choose $x \in T$, and define v by $[x, g^{-1}x] \cap [x, gx] = [x, v]$. The homeomorphism g sends $[x, v]$ into $[gx, x]$. Since it has no fixed point, we have $gv \in [v, gx]$. Then $A = \bigcup_{n \in \mathbf{Z}} g^n([v, gv])$ is the axis of g. Its embedding into T is proper, since otherwise there would be a limit point $y = \lim g^n v$ as n goes to $+\infty$ or $-\infty$. Such a point would be fixed by g.

As in the isometric case, assertion (3) is proved by induction on the number of elements of a generating system $\{g_1, \ldots, g_n\}$ of G. We assume that every $g \in G$ is elliptic, and we show that the action is trivial. Using the induction hypothesis, choose x_0 fixed by $g_1, \ldots, g_{n-1}$. Let x be the unique point such that $[x_0, x] \cap \operatorname{Fix} g_n = \{x\}$. If the action is not trivial, then there exists i with $x \notin \operatorname{Fix} g_i$. Then $\operatorname{Fix} g_i \cap \operatorname{Fix} g_n = \varnothing$, and it is easy to deduce that $g_n g_i$ is hyperbolic, a contradiction.

Define $T_{\min}$ as the union of all axes. It is shown to be a subtree as in the isometric case: if g and h are hyperbolic with disjoint axes, then gh is hyperbolic and its axis meets both the axis of g and that of h. The other properties of $T_{\min}$ are clear.

Systems of maps. We define a *finite tree* as a space homeomorphic to a simplicial tree with finitely many vertices and edges.

Now let X be a compact space homeomorphic to a finite 1-complex (in fact, X will be a finite tree, or a disjoint union of arcs or circles). Any contractible subset of X is a *subtree*. (The subtrees used below will be either open intervals or finite trees.)

We define a *system of maps* on X as a finite family $\mathscr{K} = \{\phi_i : A_i \to B_i\}_{i=1,\ldots,k}$ of homeomorphisms between subtrees of X. The system is *closed* (respectively *open*) if all subtrees A_i, B_i are closed (respectively open) in X.

A $\mathscr{K}$-*word* is a reduced word in the letters ϕ_i and ϕ_i^{-1}. We usually view it as a homeomorphism between (possibly empty) subtrees of X. The system is *non-nesting* if no $\mathscr{K}$-word takes an arc properly into itself.

An open system may be viewed as a generating set for the *pseudogroup* Γ consisting of all homeomorphisms ψ between open subsets of X such that for every x in the domain of ψ, the map ψ coincides with some $\mathscr{K}$-word on a neighbourhood of x.

Let $\mathscr{K}$ be any system of maps. Two points $x, y \in X$ are in the same *orbit* if some $\mathscr{K}$-word takes x to y. A subset of X is *invariant* if it is a union of orbits, *minimal* if it is compact, invariant, non-empty, and minimal for these three properties. Every non-empty invariant set contains a minimal set.

If Y is a subtree of X, or X itself, then a *vertex* of Y is a point v of the closure $\bar{Y}$ such that $\bar{Y}$ is not a manifold near v. An orbit of $\mathscr{K}$ is *singular* if it contains a vertex of X or of some A_i or B_i, *regular* otherwise.

A measure μ on X is *invariant* if ϕ_i takes the restriction $\mu|A_i$ onto $\mu|B_i$, for $i = 1, \ldots, k$. The *support* of μ is the complement of the largest open set with measure 0. The support of an invariant measure is compact and invariant.

The rest of the paper is divided into two parts. In Section 2 we reduce Theorem 1 to the following result.

PROPOSITION 4. *Let $\mathscr{K}$ be a non-nesting closed system of maps on a finite tree K. Assume that $\mathscr{K}$ has an infinite orbit. Then there exists a $\mathscr{K}$-invariant probability measure μ on K, with no atom.*

In Section 3 we prove Proposition 4, using Sacksteder's theorem rephrased as follows.

THEOREM 5. *A non-nesting open system of maps on S^1 with every orbit dense admits an invariant probability measure.*

2. *Trees and measures*

We prove Theorem 1, assuming Proposition 4. We start with a general construction. Let T be an $\mathbf{R}$-tree with an action of a finitely presented group G by homeomorphisms. Fix a finite presentation of G, with generators $\{g_1, \ldots, g_k\}$. Let q be the maximal length of a relation.

Choose $x_0 \in T$. Let K be the convex hull of the set of points gx_0, for g a word of length $\leqslant q$ in the generators and their inverses. It is a finite tree.

For $i = 1, \ldots, k$, let $A_i = K \cap g_i^{-1} K$ and $B_i = g_i K \cap K$, and let $\phi_i : A_i \to B_i$ be the restriction of the action of g_i. We denote by $\mathscr{K}$ the associated system of maps on K. It is closed, and it is clearly non-nesting if the action on T is.

If the action of G on T is isometric, then each ϕ_i is a partial isometry of K. In other words, it preserves the 'Lebesgue measure' induced on K by the metric of T. Conversely, we shall show how to construct an isometric action of G on an **R**-tree from a $\mathscr{K}$-invariant measure on K.

First we define a foliated 2-complex $(\bar{\Sigma}, \bar{\mathscr{F}})$ as follows (see, for example, [**7**, **5**] for similar constructions).

Consider the disjoint union of $G \times K$ and the $G \times A_i \times [0,1]$ $(i = 1, \ldots, k)$. Corresponding to each edge $g - gg_i$ in the Cayley graph of G, glue $\{g\} \times A_i \times [0,1]$ to $G \times K$ by identifying $(g, x, 0)$ with $(g, \phi_i(x))$ and $(g, x, 1)$ with (gg_i, x) for $x \in A_i$. We let $\bar{\Sigma}$ be the resulting 2-complex. We identify each $\{g\} \times K$ or $\{g\} \times A_i \times [0,1]$ with its image in $\bar{\Sigma}$, and $\{1_G\} \times K$ with K. Note that there is a natural projection from $\bar{\Sigma}$ to the Cayley graph. It is a homotopy equivalence.

The leaves of the foliation $\bar{\mathscr{F}}$ on $\bar{\Sigma}$ are defined as the classes of the smallest equivalence relation on $\bar{\Sigma}$ such that each segment $\{g\} \times \{x\} \times [0,1]$ is contained in a class (for $g \in G$, $x \in A_i$).

Left translation in G induces a free action of G on $\bar{\Sigma}$. This action preserves $\bar{\mathscr{F}}$ and defines a regular G-covering whose base is a compact foliated 2-complex $(\Sigma, \mathscr{F})$ with fundamental group free of rank k. The tree K is naturally embedded in Σ, and the intersections of leaves of $\mathscr{F}$ with K are the orbits of $\mathscr{K}$.

The inclusion of K into T extends to a G-equivariant *resolution map* $f : \bar{\Sigma} \to T$ sending every leaf of $\bar{\mathscr{F}}$ to a point: simply define $f(g, k) = gk$ for $g \in G$ and $k \in K$, and $f(g, x, t) = g\phi_i(x) = gg_i x$ for $g \in G$, $x \in A_i$, $t \in [0,1]$. Note that each $gK \subset \bar{\Sigma}$ is *taut* in the sense of [**7**]: it meets a given leaf of $\bar{\mathscr{F}}$ at most once.

Also note that $\pi_1 \bar{\Sigma}$ *is generated by (free homotopy classes of) loops contained in leaves of* $\bar{\mathscr{F}}$. This follows from the way we defined K in the beginning of this section, using finite presentability of G: the defining relations of G have length at most q, and any loop of length $\leqslant q$ in the Cayley graph may be lifted to a loop in $\bar{\Sigma}$ contained in a leaf of $\bar{\mathscr{F}}$.

Now let μ be a $\mathscr{K}$-invariant probability measure on K, with no atom. Extend μ to a G-invariant measure on $G \times K$. Since each ϕ_i preserves μ, this defines a holonomy-invariant transverse measure of $\bar{\mathscr{F}}$: the measure of a transverse arc $\{g\} \times J \times \{t\} \subset \{g\} \times A_i \times [0,1]$ is $\mu(J) = \mu(\phi_i(J))$.

We may then define the μ-length of any path γ in $\bar{\Sigma}$ as the total mass of the measure induced on γ, and the pseudo-distance $d(u, v)$ between two points of $\bar{\Sigma}$ as the infimum of the μ-lengths of paths from u to v. We define T_0 as the associated metric space, obtained by identifying u, v whenever $d(u, v) = 0$ (this happens in particular if u and v belong to the same leaf). The action of G on $\bar{\Sigma}$ induces an isometric action of G on T_0.

Following [**5**], one may also view T_0 as the metric space associated to the pseudo-distance δ defined on $G \times K$ by

$$\delta((g,x),(h,y)) = \inf\{\mu([x, x_p]) + \mu([\phi_{i_p}^{\varepsilon_p}(x_p), x_{p-1}]) + \ldots + \mu([\phi_{i_2}^{\varepsilon_2}(x_2), x_1]) + \mu([\phi_{i_1}^{\varepsilon_1}(x_1), y])\},$$

where the infimum is taken over all words $g_{i_1}^{\varepsilon_1} \ldots g_{i_p}^{\varepsilon_p}$ representing $h^{-1}g$ (with $\varepsilon_j = \pm 1$) and all points x_j in the domain of $\phi_{i_j}^{\varepsilon_j}$.

We note that T_0 *is an* **R**-*tree*, *and* $d(x,y)=\mu([x,y])$ *for* $x,y\in K$ (viewed as a subset of $\bar{\Sigma}$). These assertions are completely similar to Corollary III.5 of [**10**] and Lemma 3.3 of [**7**], respectively. They are based on the fact that $\pi_1\bar{\Sigma}$ is generated by (free homotopy classes of) loops contained in leaves of $\bar{\mathscr{F}}$, and on tautness of K. Indeed, the same arguments as in [**10**] and [**7**] apply, though μ is not assumed to have full support.

After this general construction, let us be more specific and consider a non-trivial non-nesting action of G. Recall that $\mathscr{K}$ is then non-nesting.

Suppose that the $\mathscr{K}$-orbit of some $x\in K$ is regular and finite. Since $\mathscr{K}$ is non-nesting, points near x also have finite orbits, with the same number of elements (or twice that number if some $\mathscr{K}$-word fixes x and reverses orientation).

If all orbits are finite, then all leaves of the foliation $\mathscr{F}$ on the compact 2-complex Σ are compact, and we may find a $\mathscr{K}$-invariant measure μ with full support (and no atom). The associated tree T_0 is simplicial (it is simply the space of leaves of $\bar{\mathscr{F}}$), and the map $f:\bar{\Sigma}\to T$ factors through T_0. Theorem 1 is trivial in this case.

If there is an infinite orbit, Proposition 4 provides a $\mathscr{K}$-invariant probability measure μ on K, with no atom.

Let T_0 be the **R**-tree associated to μ as before. We show that the isometric action of G on T_0 satisfies the required conditions. Unfortunately, the map $f:\bar{\Sigma}\to T$ need not factor through T_0.

To avoid confusion, we denote by $K'=\{1_G\}\times K$ the preferred copy of K in $\bar{\Sigma}$. We denote by x, x' corresponding points in K and K', and $S'\subset K'$ corresponding to the support S of μ. Note that S has no isolated point, therefore it consists of uncountably many orbits.

Let $\pi:\bar{\Sigma}\to T_0$ be the canonical projection. Since $d(x',y')=\mu([x,y])$ for $x',y'\in K'$, the restriction of π to K' is monotonous: it simply collapses each component of $K'\backslash S'$ to a point. In particular, the restriction of π to S' is finite to one, and the image $K_0=\pi(K')=\pi(S')$ is a finite tree in T_0.

Let e be an edge of K meeting some regular $\mathscr{K}$-orbit infinitely often. Let $w=\phi_{i_1}^{\varepsilon_1}\ldots\phi_{i_p}^{\varepsilon_p}$ be a $\mathscr{K}$-word defined on some non-degenerate subarc $J\subset e$ whose interior meets S, such that w maps J into e in an orientation-preserving way. The element $g_{i_1}^{\varepsilon_1}\ldots g_{i_p}^{\varepsilon_p}\in G$ acting on T_0 is hyperbolic: its translation length is positive since it equals $\mu([y,wy])$ for any $y\in J$. This proves that the action of G on T_0 is non-trivial.

We now consider arc stabilizers in T_0. Given an arc $\alpha_0\subset K_0$, we let $\alpha'=[u',v']$ be the smallest arc contained in K' that projects onto α_0 under the collapsing map $\pi|K'$. Note that u',v' are not isolated in $\alpha'\cap S'$.

Suppose the action of $g\in G$ on T_0 fixes α_0 pointwise. We claim that *the action of* g *on* T *fixes* α *pointwise* (α is the arc corresponding to α' in the subtree $K\subset T$). Because of continuity and non-nesting, it is enough to prove $gx=x$ for every $x\neq u,v$ in $S\cap\alpha$.

Since g fixes α_0, we have $d(u',gu')=d(v',gv')=0$. Choose arcs γ_u from u' to gu' and γ_v from v' to gv' whose μ-lengths are much smaller than $\mu([x,u])$ and $\mu([x,v])$. Consider the loop λ consisting of $\alpha',\gamma_v,(g\alpha')^{-1},\gamma_u^{-1}$. Arguing as in [**7**, pp. 640–641], we map a compact planar surface P into $\bar{\Sigma}$, with one boundary component going to λ and the others going into leaves of $\bar{\mathscr{F}}$. Analysing the foliation induced on P shows that x' and gx' belong to the same leaf of $\bar{\mathscr{F}}$, thus proving the claim.

Every arc in T_0 is contained in the union of finitely many images hK_0, $h\in G$. The claim then implies that any subgroup of G that fixes an arc in T_0 also fixes an arc in T.

As pointed out by G. A. Swarup, one may strengthen the statement of Theorem 1 as follows.

COROLLARY 6. *Given a finite collection of finitely generated subgroups $G_j \subset G$, each fixing a point of T, one may require that each G_j fixes a point of T_0.*

Proof. Use the above construction, replacing K by a larger finite tree. Specifically, choose $r \geqslant q$ such that each G_j may be generated by elements of length $\leqslant r$, and choose $x_j \in T$ fixed by G_j. If K contains all points gx_j, for g a word of length $\leqslant r$, then the image of x_j in T_0 is fixed by G_j.

3. *The invariant measure*

We prove Proposition 4. We perform several reductions.

3.1 *From $\mathscr{K}$ to $\mathscr{L}$.* Let $\mathscr{K} = \{\phi_i : A_i \to B_i\}_{i=1,\ldots,k}$ be a non-nesting closed system on a finite tree, with at least one infinite orbit. As was pointed out in Section 2, non-nesting forces a special structure for the set of finite regular $\mathscr{K}$-orbits. In particular, the union of all finite regular orbits is a finite union of open intervals. Its complement K_1 is a finite union of closed subtrees, not all of them points.

We disregard isolated points of K_1 and let D be the disjoint union of all closed edges of K_1. It is a multi-interval in the sense of [**6**], that is, a finite disjoint union of compact intervals. We denote by δD its set of endpoints, and $\operatorname{int} D = D \backslash \delta D$.

The system of maps $\mathscr{K}$ naturally induces a system $\mathscr{L}$ on D (compare [**6**, Parts 2 and 3]): replace each ϕ_i by the collection of its restrictions to edges of K_1, keeping only those maps whose domain contains more than a single point (we may also need to split domains at preimages of vertices of B_i). The system $\mathscr{L}$ is closed and non-nesting, and every regular orbit is infinite. Furthermore, every $\mathscr{L}$-invariant probability measure without atoms corresponds to a $\mathscr{K}$-invariant probability measure (supported on K_1). Our goal is then to construct an $\mathscr{L}$-invariant measure.

3.2 *From $\mathscr{L}$ to $\mathscr{M}$.* Let $\mathring{\mathscr{L}}$ be the open system obtained by replacing each element $\phi : A \to B$ of $\mathscr{L}$ by its restriction to the open interval $A \backslash \{\text{endpoints}\}$.

We start with the hardest case, when every minimal $\mathring{\mathscr{L}}$-invariant set is a finite singular orbit (recall that we want to apply Theorem 5, which requires dense orbits). There are then finitely many minimal sets, among which are the endpoints $d \in \delta D$. By splitting D (thus changing $\mathscr{L}$ and $\mathring{\mathscr{L}}$), we may assume that the points of δD are the only minimal sets of $\mathring{\mathscr{L}}$. Note that the closure of every $\mathring{\mathscr{L}}$-orbit contains an endpoint of D.

For every $d \in \delta D$, we choose an $\mathscr{L}$-word w_d defined on a non-degenerate closed interval I_d containing d, and sending d to a point $\tilde{d} \in \operatorname{int} D$; such a word exists because otherwise non-nesting of $\mathscr{L}$ would imply finiteness of orbits near d.

Let E be the set of points $p \in \delta D$ such that some $\mathring{\mathscr{L}}$-orbit accumulates on p. For every $p \in E$, we choose points $x_p, y_p \in I_p$ such that some $\mathring{\mathscr{L}}$-word τ_p sends x_p to y_p in an orientation-preserving way. Since every $\mathring{\mathscr{L}}$-orbit in $\operatorname{int} D$ accumulates on E, we may require, furthermore, that for every $d \in \delta D$, the orbit of $\tilde{d}$ meets the union of all intervals (x_p, y_p), $p \in E$. After changing w_d, we may assume that every $\tilde{d}$ belongs to some (x_p, y_p). Note that every $\mathring{\mathscr{L}}$-orbit in $\operatorname{int} D$ meets some (x_p, y_p) infinitely often.

For $p \in E$, we define C_p to be the circle obtained by identifying the endpoints of $[x_p, y_p]$, and we let C be the disjoint union of these circles. Our next goal is to construct a non-nesting open system of maps $\mathscr{M}$ on C with *every* orbit infinite, in such a way that an $\mathscr{M}$-invariant measure on C provides an $\mathscr{L}$-invariant measure on D. (To be precise: the pseudogroup generated by $\mathscr{M}$ on C will be equivalent in the sense of Haefliger [**9**] to the pseudogroup generated by $\mathring{\mathscr{L}}$ on $\operatorname{int} D$.)

First we need to construct a finite family $\mathscr{P}$ of homeomorphisms $\gamma: U \to V$ between open intervals $U \subset \operatorname{int} D$ and intervals $V \subset C$. It will consist of three types of maps.

For each $p \in E$, we include the natural map π_p from (x_p, y_p) to C_p. Then, for each p, we consider the points x_p, y_p and their common image z_p in C. Restrict τ_p to a homeomorphism between a small neighbourhood X_p of x_p and a neighbourhood Y_p of y_p. Since τ_p preserves orientation, both these neighbourhoods are naturally homeomorphic to a neighbourhood Z_p of z_p in C_p. We include the maps $X_p \to Z_p$ and $Y_p \to Z_p$.

The ranges of the maps constructed so far cover C. If the domains cover $\operatorname{int} D$, then we may stop. Otherwise, we need a third type of maps. Let $x \in \operatorname{int} D$. Some $\mathring{\mathscr{L}}$-word α_x sends a neighbourhood U_x of x to an open interval V_x contained in some (x_p, y_p). The word α_x may be chosen to be constant near each endpoint of D. By compactness, we deduce that $\operatorname{int} D$ may be covered by finitely many U_x. We include the corresponding finite set of maps from U_x to $\pi_p(V_x) \subset C_p$.

This defines the family $\mathscr{P}$. We now use it to carry $\mathring{\mathscr{L}}$ over to a system $\mathscr{M}$ on C. Let $\gamma_1: U_1 \to V_1$ and $\gamma_2: U_2 \to V_2$ be two elements of $\mathscr{P}$. If $U_1 \cap U_2 \neq \varnothing$, then $\gamma_2 \gamma_1^{-1}$ is a homeomorphism between non-empty subintervals of C. We include it in $\mathscr{M}$. Similarly, for $\theta \in \mathring{\mathscr{L}}$, we include $\gamma_2 \theta \gamma_1^{-1}$ if its domain is non-empty. We define $\mathscr{M}$ as the set of all maps thus obtained, for all possible choices of γ_1, γ_2 in $\mathscr{P}$ and θ in $\mathring{\mathscr{L}}$. It is an open system of maps on C.

The construction of $\mathscr{P}$ and $\mathscr{M}$ was done in such a way that the following properties hold: given γ_1, γ_2 in $\mathscr{P}$ and an $\mathring{\mathscr{L}}$-word w, given x in the domain of $\gamma_2 w \gamma_1^{-1}$, there is an $\mathscr{M}$-word equal to $\gamma_2 w \gamma_1^{-1}$ on a neighbourhood of x; similarly, given an $\mathscr{M}$-word w, and y in the domain of $\gamma_2^{-1} w \gamma_1$, some $\mathring{\mathscr{L}}$-word coincides with $\gamma_2^{-1} w \gamma_1$ near y.

Every $\mathscr{M}$-orbit is infinite, because every $\mathring{\mathscr{L}}$-orbit in $\operatorname{int} D$ meets some (x_p, y_p) infinitely often. Also note that $\mathscr{M}$ is non-nesting because every orientation-preserving $\mathscr{M}$-word that has a fixed point is a restriction of the identity. Furthermore, any $\mathscr{M}$-invariant probability measure with no atom lifts to an $\mathring{\mathscr{L}}$-invariant measure on $\operatorname{int} D$. The existence of the words w_a (which depends on non-nesting of $\mathscr{L}$) implies that this measure has finite total mass, hence extends to an $\mathscr{L}$-invariant measure on D.

We have now reduced our problem to finding an $\mathscr{M}$-invariant measure with no atom (assuming that every minimal set of $\mathring{\mathscr{L}}$ is infinite).

3.3 *From $\mathscr{M}$ to $\mathscr{N}$.* Let $F \subset C$ be a minimal set of $\mathscr{M}$. It is infinite and has no isolated point. For each component C_p of C that meets F, we choose a collapsing map $\rho_p: C_p \to C'_p$, where C'_p is another circle and ρ_p sends each component of $C_p \setminus (F \cap C_p)$ to a point. Let C' be the union of the circles C'_p.

Given an element $\phi: A \to B$ of $\mathscr{M}$ whose domain meets F, we consider the images A', B' of A, B in C', and the natural homeomorphism ϕ' between the interiors of A' and B' (note that A', B' are non-degenerate intervals, but they need not be open). We let $\mathscr{N}$ be the collection of these ϕ', an open system of maps on C'. It is non-nesting and every regular orbit is dense, because every $\mathscr{M}$-orbit contained in F is dense in F, but there may be finite singular orbits.

3.4 *From $\mathcal{N}$ to $\mathcal{O}$.* Choose an interval $[x, y]$ disjoint from all finite singular orbits, such that some orientation-preserving $\mathcal{N}$-word sends x to y. Perform the same operation as in 3.2, so as to obtain a system $\mathcal{O}$ on the circle obtained by identifying the endpoints of $[x, y]$. This last system is open, non-nesting, with every orbit dense. By Sacksteder's theorem, it admits an invariant measure. This measure lifts first to an $\mathcal{N}$-invariant measure, then to the required $\mathcal{M}$-invariant measure.

This completes the proof when every minimal set of the original system $\mathring{\mathcal{L}}$ is finite. If $\mathring{\mathcal{L}}$ has an infinite minimal set F, we first collapse to a point every component of $D\backslash F$ (as in 3.3). We obtain a system $\mathcal{N}$ on a multi-interval with every regular orbit dense, and we deal with it as before.

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