Deformations of Length Functions in Groups

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In this paper we study the space of actions of a group on $\mathbf{R}$-trees. An $\mathbf{R}$-*tree* is a metric space (T, d), such that any two distinct points x, y are joined by a unique arc $[x, y]$ and every arc is isometric to a segment in $\mathbf{R}$. Let G act on both T and T'. It is understood that an action is always by isometries. A *morphism* from a tree T to a tree T' is an equivariant map $\phi : T \to T'$, such that for each segment $[x, y]$ there is a segment $[x, w] \subseteq [x, y]$, such that $\phi \,|\, [x, w]$ is an isometry.

Let $\phi : T \to T'$ be a morphism. We will show that ϕ may be continuously deformed to the identity. In particular, we define morphisms $\phi_{st} : T_s \to T_t$, $0 \leq s \leq t \leq 1$, such that $\phi_{01} = \phi$ and $\phi_{st} \circ \phi_{rs} = \phi_{rt}$. Then here is our main result.

4.8 Theorem. *Let G be a group; $\mathcal{X}$ be the space of all actions of G on $\mathbf{R}$-trees; and $\mathcal{C}(\mathcal{X})$ the space of all morphisms between elements of $\mathcal{X}$. Then the function*

$$\mathcal{C}(\mathcal{X}) \times \{(s,t) | 0 \leq s \leq t \leq 1\} \to \mathcal{C}(\mathcal{X})$$

defined as $(\phi, (s,t)) \mapsto \phi_{st}$ is continuous.

A morphism $\phi : T \to T'$ has both properties of infiniteness and finiteness. It may happen that the pre-image of every point is infinite and unbounded. However, it readily follows from the definition that for any segment σ in T, ϕ folds at at most finitely many points on σ. A morphism $\phi : T \to T_0$ *folds* at a point $x \in T$ if there are segments $[x, y]$ and $[x, y']$, such that $[x, y] \cap [x, y'] = \{x\}$; $\phi|[x, y]$ and $\phi|[x, y']$ are embeddings; and $\phi([x, y]) = \phi([x, y'])$. In particular, the image of each segment is a finite simplicial tree. To prove Theorem 4.8 we need to correctly deform the $\mathbf{R}$-trees – the deformation of the morphisms is free. We essentially deform by folding and we take the philosophy that folding is a local process. It suffices to understand how to fold a segment into a finite simplicial tree. Our deformation is very explicit.

We have the following applications.

6.3 Theorem. *Let F_n be a free group of rank n and $\mathcal{X}$ be the space of all non-trivial semi-simple actions of F_n on $\mathbf{R}$-trees. Then $\mathcal{X}$ is contractible.*

The idea in the proof of Theorem 6.3 is to construct a retract of sorts. F_n is the fundamental group of a simple space – the wedge of n circles. Let $\mathcal{G}$ be the space of actions on $\mathbf{R}$-trees which are dual to measured laminations on the wedge of n circles. This space is easily seen to be homeomorphic

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to $(n-1)$-simplex times $\mathbf{R}^+$ which is contractible. To each element T' in $\mathcal{X}$ we continuously associate a morphism from an element T of $\mathcal{G}$ to T'. By applying Theorem 4.8 one deforms T to T'.

The deformation is controlled enough to conclude that certain important subspaces of $\mathcal{X}$ are also contractible. In particular we give new proofs that both Culler-Vogtmann space and its closure are contractible.

Here is our second application.

7.3 Theorem. *Let F be a closed hyperbolic surface and $\mathcal{X}$ be the space of all non-trivial semi-simple actions of $\pi_1 F$ on* $\mathbf{R}$*-trees. Then $\mathcal{X}$ strong deformation retracts to a sphere of dimension $-3\chi(F)$.*

The idea in the proof of Theorem 7.3 is similar to above. Only this time $\mathcal{G}$ is the space of actions on $\mathbf{R}$-trees which are dual to measured geodesic laminations on F. This space is homeomorphic to $S^n \times \mathbf{R}^+$, where $n = -3\chi(F)$ [Th].

This paper extends the ideas of Michael Steiner [St2]. He considered the space of actions of the free group of rank n on $\mathbf{R}$-trees. He used two facts about this group. It has a free product factorization with factors $\mathbf{Z}$; and the actions of $\mathbf{Z}$ are classified. The actions of the factors are in a certain sence independant of each other. He then used [St1] to deform all actions to a standard action. He also obtained some of our results in §6.

M. Culler and J. W. Morgan [Cu-Mo] asked what the space of non-trivial actions on $\mathbf{R}$-trees was for an arbitrary group. Theorems 6.3 and 7.3 answer that question upto homotopy for the free group and the fundamental group of a closed hyperbolic surface. The methods of §§6 and 7 work for any group which has a corredsponding space $\mathcal{G}$ to which Lemmas like 6.2 and 7.2 applies. It is an obvious question whether in general $\mathcal{PLF}(G)$ strong deformation retracts to $\mathcal{G}$, where $\mathcal{G}$ is the space of actions of G on $\mathbf{R}$-trees which are dual to measured laminations on some finite complex with fundamental group G?

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1. Definitions.

The definitions below may also be found in [Mo-Sh1], [Cu-Mo] and [A-B]. The following equivalent definition of $\mathbf{R}$-tree will be useful later. Let (T,d) be a metric space. Define $[x,y] = \{z | d(x,z) + d(z,y) = d(x,y)\}$. An *$\mathbf{R}$-tree* is a non-empty, metric space (T,d) satisfying

(i) for all $x, y \in T$, $[x,y]$ is isometric to a segment in R;

(ii) for all $x,y,z \in T$, if $[x,y] \cap [y,z] = \{y\}$, then $[x,z] = [x,y] \cup [y,z]$; and

(iii) for all $x,y,z \in T$, there is a $w \in T$, such that $[x,y] \cap [y,z] = [y,w]$.

Given an $\mathbf{R}$-tree T and $x \in T$, define $B_x = \{[x,y] \,|\, y \in T - \{x\}\}$. Define an equivalence relation by $[x,y] \sim [x,z]$ if $[x,y] \cap [x,z] = [x,w]$, for some $w \in T - \{x\}$. A *direction* at x is an equivalence class in B_x.

Let $x \in T$. If x has exactly two directions, then say x is an *edge*, otherwise x is a *vertex*. Notice that x is an edge if and only if $T - \{x\}$ has exactly two connected components.

In this paper an R-tree T always comes with a group G acting on the left by isometries. We will denote the tree together with the action simply as T.

The action $G \times T \to T$ is *trivial* if there is a fixed point. The action is *minimal* if there is no invariant proper subtree. The action is *reducible* if either

(i) every element of G fixeds a point of T;

(ii) G fixes exactly one end of T; or

(iii) G leaves a set of two ends of T invariant.

Otherwise, the action is *irreducible*. An action of type (iii) is a *shift* if it fixes each end and *dihedral* if it interchanges the two ends. The action is *semi-simple* if it is either irreducible; trivial; or is reducible of type (iii) above.

Given $G \times T \to T$ define its *length function* $\ell : G \to [0, +\infty)$ by $\ell(g) = \ell_T(g) = \min\{x \in T | d(x, g(x))\}$. The *characteristic set* of g is $T_g = \{x | d(x, g(x)) = \ell(g)\}$. Given a base point $x_* \in T$, the *Lyndon length function* [Ly] is $\mathcal{L}(g) = d(x_*, g(x_*))$. We will work with the length function as opposed to the Lyndon length function.

The *space of length functions* $\mathcal{LF}(G) \subset [0, +\infty)^G$ is the set of all length functions. This space is usually not compact, so it is preferable to work with the following. The *space of projective length functions* $\mathcal{PLF}(G)$ is the image of $\mathcal{LF}(G) - 0$ in $([0, +\infty)^G - 0)/\mathrm{R}^+$. If G is finitely generated, then $\mathcal{PLF}(G)$ is compact [Cu-Mo], [Pau].

A *morphism* from a tree T to a tree T' is an equivariant map $\phi : T \to T'$, such that for each segment $[x, y]$ there is a segment $[x, w] \subseteq [x, y]$, such that $\phi \,|\, [x, w]$ is an isometry. The morphism *folds* at a point $x \in T$ if there are segments $[x, y]$ and $[x, y']$, such that $[x, y] \cap [x, y'] = \{x\}$; $\phi|[x, y]$ and $\phi|[x, y']$ are embeddings; and $\phi([x, y]) = \phi([x, y'])$. A morphism either is a monomorphism or folds at some point [Mo-Ot].

2. Topology.

M. Gromov [Gr] defined a topology on a set of metric spaces. This was generalized independantly by both Paulin [Pau1] and K. Fukaya [Fu] to an equivariant topology on a set of metric spaces which a group acts on. The topology of Paulin is finer. In this section we follow [Pau1] and define a topology on a set of equivariant maps between metric spaces.

We begin by reviewing. Let X, Y be metric spaces. An *ϵ-approximation* from X to Y is a relation $R \subseteq X \times Y$, such that R is onto both X and Y; and $x_0 R y_0$, $x_1 R y_1$ implies $|d(x_0, x_1) - d(y_0, y_1)| < \epsilon$. (The relation R is *onto* if its image is onto under projection to each of X and Y.)

Let G act on both X and Y. Let $P \subseteq G$, $K \subseteq X$ and $L \subseteq Y$. An ϵ-approximation R from K to L is *P-equivariant* if $\alpha \in P$, $x \in K$, $\alpha x \in K, y \in L$ and xRy implies $\alpha y \in L$ and $\alpha x R \alpha y$. Notice that if there is a P-equivariant, ϵ-approximation from X to Y and a P-equivariant, η-approximation from Y to Z, then there is an P-equivariant, $(\epsilon + \eta)$-approximation from X to Z.

This allows us to define a topology on a set of G-metric spaces. Fix a group G and $\mathcal{X}$ be a set of G-metric spaces. Given $X \in \mathcal{X}$, K a compact subset of X, P a finite subset of G and $\epsilon > 0$, define the *basic open set* $U(X, K, P, \epsilon)$ to be the set of all $Y \in \mathcal{X}$, such that for some compact $L \subseteq Y$ there is a P-equivariant, closed ϵ-approximation from K to L. (The relation R is *closed* if it is closed as a subspace of $X \times Y$.) Let $\mathcal{X}$ have the topology generated by all basic open sets. These basic open sets have the following nice property. If $P_0 \subseteq P, K_0 \subseteq K$ and $\epsilon_0 \geq \epsilon$, then $U(X, K, P, \epsilon) \subseteq U(X, K_0, P_0, \epsilon_0)$. Notice that in general the topology is *not* Hausdorff.

We now generalize the above.

2.1 Definition. Let $\phi : X \to X', \psi : Y \to Y'$ be maps. An *ϵ-approximation* from $\phi : X \to X'$ to $\psi : Y \to Y'$ is a pair of relations (R, R'), such that

(i) R and R' are ϵ-approximations from X to Y and from X' to Y', respectively; and

(ii) xRy implies $\phi(x)R'\psi(y)$.

Define a metric on $X \times X'$ as $d((x_0, x_0'), (x_1, x_1')) = \max\{d(x_0, x_1), d(x_0', x_1')\}$. Similarly define a metric on $Y \times Y'$. We will write $(x, x')(R, R')(y, y')$, whenever xRy and $x'R'y'$. Notice that (R, R') is an ϵ-approximation from $X \times X'$ to $Y \times Y'$.

2.2 Definition. Let $\phi : X \to X', \psi : Y \to Y'$ be equivariant maps between G-metric space. Let $P \subseteq G$, $K \times K' \subseteq X \times X'$ and $L \times L' \subseteq Y \times Y'$. An ϵ-approximation (R, R') from $\phi : K \to K'$ to $\psi : L \to L'$ is *P-equivariant* if both R and R' are P-equivariant.

Now let $\mathcal{C}$ be a set of equivariant maps between G-metric spaces. Let $\phi : X \to X'$ be in $\mathcal{C}$. Given $K \times K'$ a compact subset of $X \times X'$, P a finite subset of G and $\epsilon > 0$, define the *basic open set* $U(\phi, K \times K', P, \epsilon)$ to be the set of all maps $\psi : Y \to Y'$ in $\mathcal{C}$, such that for some compact $L \times L' \subseteq Y \times Y'$ there is a P-equivariant, closed ϵ-approximation from $\phi : K \to K'$ to $\psi : L \to L'$. Let $\mathcal{C}$ have the topology generated by all basic open sets.

Given a function $f : A \to A'$, define $\mathcal{D}(f) = A, \mathcal{R}(f) = A'$. The following propositions are obvious.

2.3 Proposition. *Let G be a group and $\mathcal{X}$ be a space of G-metric spaces and let $\mathcal{C}(\mathcal{X})$ be the space of equivariant maps between elements of $\mathcal{X}$. Then the function $\mathcal{C}(\mathcal{X}) \to \mathcal{X}$ defined by $\phi \mapsto \mathcal{D}(\phi)$ is continuous.* ∎

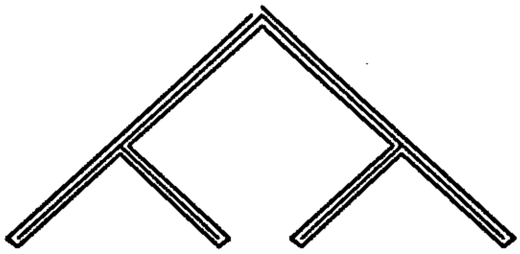
2.4 Proposition. *Let G be a group and $\mathcal{X}$ be a space of G-metric spaces and let $\mathcal{C}(\mathcal{X})$ be the space of equivariant maps between elements of $\mathcal{X}$. Then the function $\mathcal{C}(\mathcal{X}) \to \mathcal{X}$ defined by $\phi \mapsto \mathcal{R}(\phi)$ is continuous.* ∎

2.5 Proposition. *Let G be a group and $\mathcal{X}$ be a space of G-metric spaces and let $\mathcal{C}(\mathcal{X})$ be the space of equivariant maps between elements of $\mathcal{X}$. Then the function $\mathcal{X} \to \mathcal{C}(\mathcal{X})$ defined by $X \mapsto Id_X$ is an embedding.* ∎

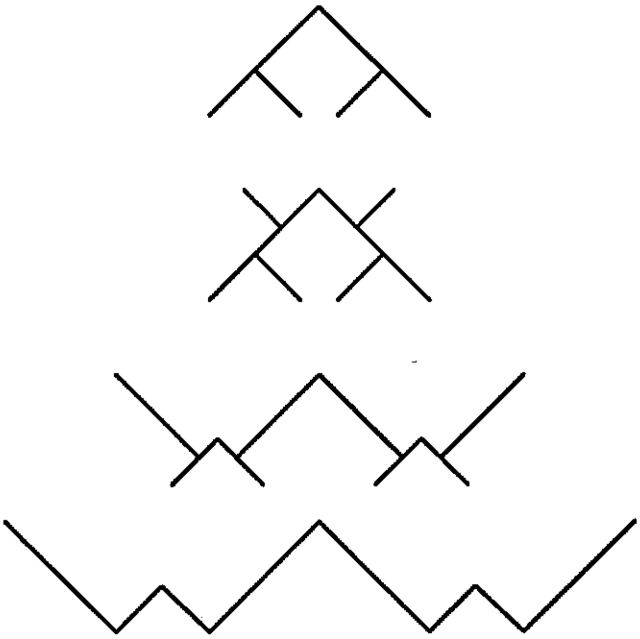
3. A Canonical Tree.

For this section fix a group G and a morphism $\phi : T \to T'$ between two $\mathbf{R}$-trees which G acts on. We show there is a canonical deformation of T to T' in the following sense. It is proved that there are morphisms between $\mathbf{R}$-trees $\phi_{st} : T_s \to T_t$, $0 \le s \le t \le 1$, such that $\phi_{01} = \phi$ and $\phi_{st} \circ \phi_{rs} = \phi_{rt}$. In the next section we show ϕ_{st} varies continuously with ϕ and (s,t).

In particular we will deform T to T'. This deformation may be complicated. However, it will suffice to understand how to deform a segment to a finite simplicial tree. See Figures 1a. and 1b.



An epi-morphism from a segment to a finite simplicial tree
Figure 1a.



A deformation from a segment to a finite simplicial tree
Figure 1b.

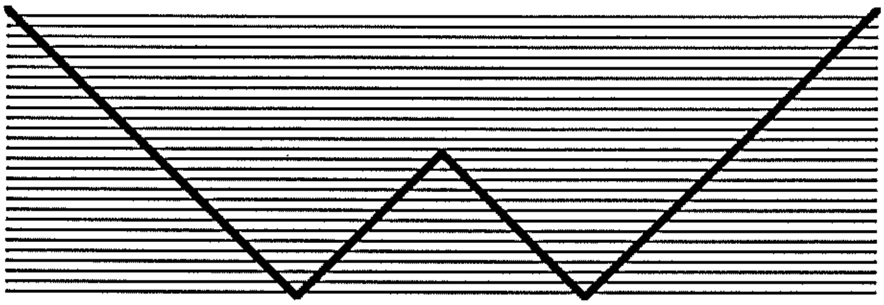
Define $W = W(\phi) = T \times T'$. Let $\mathcal{F}$ be a decomposition whose elements are the sets $T \times \{y\}$,

$y \in T'$. Clearly $W/\mathcal{F} = T'$.

More generally, define $W_t = W_t(\phi) = \{(x,x') \in W | d(\phi(x),x') \leq t\}$, $0 \leq t \leq 1$. Notice that W_t contains the graph of ϕ, for all t. Let $\mathcal{F}_t$ a decomposition of W_t whose elements are the path components of $W_t \cap (T \times \{y\})$, $y \in T'$. Let T_t be the quotient $W_t/\mathcal{F}_t$. Since the decomposition is equivariant, G acts on T_t. The quotient T_t has the quotient topology, but we will not make use of it. We will directly define a metric on T_t which makes it an R-tree.

Let $\nabla = \{(s,t) | 0 \leq s \leq t \leq 1\}$. Clearly for all $(s,t) \in \nabla$, the inclusion $W_s \to W_t$ induces a map $\phi_{st} : T_s \to T_t$. Notice that $\phi_{01} = \phi$ and $\phi_{st} \circ \phi_{rs} = \phi_{rt}$.

It will be shown that this map is a morphism between trees. Figure 2 is a picture of $W(\phi)$ for ϕ a morphism between segments T, T'.



$W(\phi)$
Figure 2

The first two lemmas follow from the fact that W_t is a strong deformation retract of W.

3.1 Lemma. *The space W_t is connected and simply connected.* ∎

3.2 Lemma. *The space W_t is a closed subset of W.* ∎

We now use an idea of H. Gillet and P. Shalen [G-S]. They found sufficient conditions for a measured foliation on a singular surface to be dual to an R-tree. Their ideas translate routinely for the decompositon of W_t.

Say a path $\gamma : [0,1] \to W_t$ is *taut* if $\gamma^{-1}(\ell)$ is connected for each component ℓ of $\mathcal{F}_t$. Say two taut paths γ, γ' are *equivalent* if there is a 1-parameter family of taut paths γ_r, such that $\gamma_0 = \gamma$; $\gamma_1 = \gamma'$; and $\gamma(s), \gamma_r(s)$ lie in the same element of $\mathcal{F}_t$, for all r and $s = 0, 1$.

It will simplify subsequent proofs if we use the following. Say a map f from a simplex to W_t is *linear* if there are two segments $\sigma \subseteq T$, $\sigma' \subseteq T'$, such that the image of f lies in $(\sigma \times \sigma') \cap W_t$ and f as a function into $\sigma \times \sigma'$ is linear. In the following it will suffice to consider only piecewise-linear paths. Recall that $\gamma\gamma'$ is the path product of γ and γ'.

3.3 Lemma. *Let $\gamma = \gamma_1 \cdots \gamma_n$, be a path in W_t, such that each γ_i is linear. If γ is not taut, then for* *some $k < l$, $\gamma_k \cdots \gamma_l$ is not taut and $\gamma_{k+1} \cdots \gamma_{l-1}$ lies in a single element of $\mathcal{F}_t$.*

Proof. It suffices to consider only the case that $\gamma(0), \gamma(1)$ lie in the same element of $\mathcal{F}_t$. By Lemma 3.1 there is a homotopy $H_s : [0,1] \to W_t$, such that $H_0 = \gamma$; $H_s(0) = \gamma(0), H_s(1) = \gamma(1)$, for all s; and $H_1([0,1])$ lies in a single elenent of $\mathcal{F}_t$. We may assume that H is piecewise-linear.

So for each element $\ell \in \mathcal{F}_t$, $H^{-1}(\ell)$ is a finite 2-complex in $[0,1] \times [0,1]$. In the special case that $H^{-1}(\ell)$ is 1-dimensional for all $\ell \in \mathcal{F}_t$, the result follows from Euler characteristic considerations.

In the more general case, the result follows similarly. ∎

3.4 Lemma. *Let $\gamma = \gamma_1\gamma_2\gamma_3$ be a path in W_t from b_0 to b_1, such that each γ_i is linear and γ_2 lies in a single element of $\mathcal{F}_t$. Then there exists paths γ_1', γ_1'', γ_3', γ_3'', and γ_2', such that $\gamma_1 = \gamma_1'\gamma_1''$; $\gamma_3 = \gamma_3''\gamma_3'$; $\gamma_1'', (\gamma_3'')^{-1}$ are equivalent; γ_2' lies in a single element of $\mathcal{F}_t$; and $\gamma_1'\gamma_2'\gamma_3'$ is taut.*

Proof. This follows from Lemma 3.2. ∎

3.5 Lemma. *For any two points $b_0, b_1 \in W_t$ there is a piecewise-linear taut path from b_0 to b_1. Futhermore, if γ, γ' are piecewise-linear taut paths from b_0 to b_1, then γ and γ' are equivalent.*

Proof. Let $b_0, b_1 \in W_t$. Then there is a piecewise linear taut path γ from b_0 to b_1 which is a product of a path from b_0 to the graph of ϕ; a path along the graph; and a path from the graph to b_1.

If γ is taut, we are done. If γ is not taut, then write $\gamma = \gamma_1 \cdots \gamma_n$, where each γ_i is linear and therefore taut. By Lemma 3.3 for some $k < l$, $\gamma_k \cdots \gamma_l$ is not taut and $\gamma_{k+1} \cdots \gamma_{l-1}$ lies in a single element of $\mathcal{F}_t$. Now appling Lemma 3.4 to $\gamma_k(\gamma_{k+1} \cdots \gamma_{l-1})\gamma_l$, we can replace $\gamma_k \cdots \gamma_l$ by a taut path. By induction on n we produce a taut path from b_0 to b_1.

Now suppose γ, γ' are piecewise-linear taut paths from b_0 to b_1. Then $\gamma'\gamma^{-1}$ is a piecewise-linear path from b_0 to b_0. If γ, γ' is taut, then b_0, b_1 lie on the same element of $\mathcal{F}_t$ and γ, γ' are equivalent. Otherwise, we can apply Lemmas 3.3 and 3.4 to show again γ, γ' are equivalent. ∎

Let $\pi : T_t \to T'$ be induced by the inclusion $W_t \to W$. Define $d : T_t \times T_t \to \mathbf{R}$ by $d([x_0], [x_1]) = length(\pi \circ \gamma)$, where γ is a piecewise-linear taut path from x_0 to x_1. Lemma 3.5 implies d is well defined and Lemmas 3.3 and 3.4 imply it is a metric.

3.6 Lemma. *(T_t, d) is an* R*-tree.*

Proof. We only need to show that (T_t, d) satisfies the three axioms for an R-tree. This follows from Lemmas 3.3 and 3.4. ∎

Lemma 3.6 essentially tells us that a certain quotient of T is an R-tree. It is unknown in general when the quotient of an R-tree is an R-tree.

3.7 Lemma. *For all* $(s,t) \in \nabla$, $\phi_{st} : T_s \to T_t$ *is a morphism.*

Proof. Let $[x,y]$ be a segment in T_s. Then $[x,y]$ is the image of a piecewise linear taut path in W_s. So there is some segment $[x,w]$ which is the image of a linear taut path. Clearly $\phi_{st}|[x,w]$ is an isometry. ■

We should warn the reader of two properties of our deformation. Firstly, it may happen that ϕ is an epi-morphism, but on some ϕ_{st} it is not an epi-morphism. See Figure 2. Secondly, it may happen that the actions on both T, T' are minimal, but the action on some T_t is not minimal. This causes a minor problem which will be addressed later.

However, the deformation has some nice properties. We record one here for use in §§6 and 7.

3.8 Proposition. *Let* G *act on* T *and* T' *and let* $\phi : T \to T'$ *be a morphism. If the actions on* T, T' *are semi-simple, then the action on* T_t *is semi-simple, for all* $0 \leq t \leq 1$.

Proof. Consider the case that the action on T is irreducible. We will show that the action on T_t is irreducible, for all $0 \leq t < 1$. Let a be some group element that acts as translation on T'. Since the action on T is irreducible, there is another group element b conjugate to a which has an axis in T disjoint with the axis of a in T; and which necessarily also acts by translation on T'. By definition of T_t the axes of a and b in T_t, $0 \leq t < 1$ intersect in a finite segment. Thus the action is irreducible.

The cases that T is trivial or reducible of type (iii) are easy to prove. ■

Another nice property is that $\{\phi_{st}\}_{0\leq s\leq t<1}$ converges strongly and its limit is $T_1 = T'$. (See [G-S] for the definitions.) We will not use this.

4. A Canonical Deformation.

This section is a continuation of §3. We show that $\phi_{st} : T_s \to T_t$ varies continuously with ϕ and (s,t). The proof is intuitive, but technical. Here is an over view.

We want to show that if $(\phi,(s,t))$ and $(\psi,(p,q))$ are close, then ϕ_{st} and ψ_{pq} are close. Roughly we show that a relation from ϕ to ψ induces a relation from ϕ_{st} to ψ_{pq}. Since we measure distances with taut paths, we show that a taut path in $W_t(\phi)$ is close to a taut path in $W_q(\psi)$.

The first lemma is a strengthening of the first half of Lemma 3.5. Say a path γ in $W_t(\phi)$ is *very linear* if γ is linear and the image lies in $\sigma \times \sigma'$, where $\phi|\sigma$ is linear.

4.1 Lemma. *Let* $\phi : K \to K'$ *be a morphism with* K, K' *finite simplicial trees. Then there exists* N *such that for all* $z_0, z_1 \in W_q(\phi)$, *there is a taut path* $\gamma = \gamma_1 \cdots \gamma_N$ *in* $W_t(\phi)$ *from* z_0 *to* z_1 *with each* γ_i *very linear.*

Proof. Take N to be $3(3+M)$, where M is the number of points in T at which ϕ folds. The rest of the proof is like the proof of Lemma 3.5. ■

We need some notation. If $z \in W_t$, then let $[z] = [z]_t$ be its image in T_t. If R is a ϵ-approximation from K to L, then let R_η be an open η-neighborhood of R in $K \times L$. It is easy to see that R_η is an $(\epsilon + 2\eta)$-approximation form K to L.

4.2 Lemma. *For all $\epsilon > 0$ there is a $\delta > 0$, such that if (R, R') is a δ-approximation from $\phi : K \to K'$ to $\psi : L \to L'$ and $|\frac{t}{1-t} - \frac{q}{1-q}| < \delta$, then there is an ϵ-approximation (S, S') from $\phi : K \to K'$ to $\psi : L \to L'$, such that for all $z \in W_t(\phi)$ there is a $w \in W_q(\psi)$, such that $z(S, S')w$.*

Proof. Let ϵ be given. Take $\delta = \epsilon/3$. Set $(S, S') = (R_\delta, R'_\delta)$. ∎

Let $\phi : K \to K'$ and $\psi : L \to L'$ be morphisms and let γ, ζ be piecewise linear paths in $W(\phi)$, $W(\psi)$, respectively. Say that γ, ζ are *ϵ-close* if there is an ϵ-approximation from $\gamma : [0, 1] \to K \times K'$ to $\zeta : [0, 1] \to L \times L'$; and $length(\pi \circ \gamma|[t_0, t_1]) = length(\pi \circ \zeta|[t_0, t_1])$, for all $0 \le t_0 \le t_1 \le 1$.

4.3 Lemma. *For all $\epsilon > 0$ there is a $\delta > 0$, such that if (R, R') is a δ-approximation from $\phi : K \to K'$ to $\psi : L \to L'$; $0 \le t, q \le 1$; $z_i \in W_t(\phi)$, $w_i \in W_q(\psi)$, such that $z_i(R, R')w_i$, for $i = 0, 1$; and γ is a very linear path in $W_t(\phi)$ from z_0 to z_1, then there is a piecewise linear taut path ζ in $W_q(\psi)$ from w_0 to w_1 and γ, ζ are ϵ-close.*

Proof. Let ϵ be given. Take $\delta = \epsilon/3$. Suppose (R, R') is an δ-approximation from $\phi : K \to K'$ to $\psi : L \to L'$. Also suppose $z_i \in W_t(\phi)$, $w_i \in W_q(\psi)$, such that $z_i(R, R')w_i$, $i = 0, 1$; and γ is a very linear path in $W_t(\phi)$ from z_0 to z_1.

Let $z_i = (x_i, x'_i)$ and $w_i = (y_i, y'_i)$, $i = 0, 1$. By definition $|d(x_0, x_1) - d(y_0, y_1)| < \delta$ and $|d(x'_0, x'_1) - d(y'_0, y'_1)| < \delta$. Let η_0 be the linear path in $W_1(\psi)$ from w_0 to w_1. Clearly $(Id, (R_\delta, R'_\delta))$ is a ϵ-approximation from $\gamma : [0, 1] \to K \times K'$ to $\zeta_0 : [0, 1] \to L \times L'$ and γ, ζ_0 are ϵ-close.

Also by definition $|d(\phi(x_0), \phi(x_1)) - d(\psi(y_0), \psi(y_1))| < \delta$. Since $d(x_0, x_1) = d(\phi(x_0), \phi(x_1))$, it follows that $|d(y_0, y_1) - d(\psi(y_0), \psi(y_1))| < 2\delta$. So there is a piecewise linear taut path ζ in $W_q(\psi)$ from w_0 to w_1 and $(Id, (R_\delta, R'_\delta))$ is a ϵ-approximation from $\gamma : [0, 1] \to K \times K'$ to $\zeta : [0, 1] \to L \times L'$ and γ, ζ are ϵ-close. ∎

4.4 Lemma. *For all $\epsilon > 0$ there is $\delta > 0$, such that if (R, R') is a δ-approximation from $\phi : K \to K'$ to $\psi : L \to L'$ and $|\frac{t}{1-t} - \frac{q}{1-q}| < \delta$, then for every $\gamma = \gamma_1 \cdots \gamma_N$ in $W_t(\phi)$ from z_0 to z_1 with each γ_i very linear, and $w_i \in W_q(\psi)$, such that $z_i(R, R')w_i$, for $i = 0, 1$ there is $\zeta = \zeta_1 \cdots \zeta_N$ in $W_q(\psi)$ from w_o to w_1 with each ζ_i piecewise linear taut, such that γ_i, ζ_i are ϵ-close, for all $i = 1, \ldots, N$.*

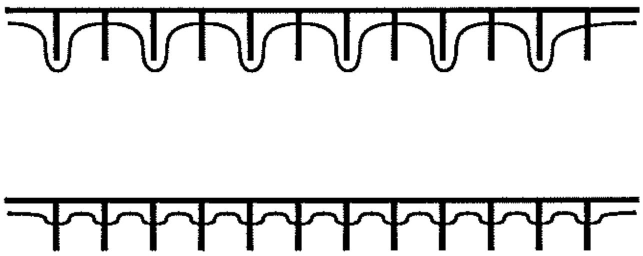
Proof. Let $\epsilon > 0$ be given. Set $\epsilon_1 = \epsilon$ and take the pair ϵ_1, δ_1 as in Lemma 4.3. Set $\epsilon_2 = \delta_1$ and take the pair ϵ_2, δ_2 as in Lemma 4.2. Take $\delta = \delta_2$.

Suppose (R, R') is a δ-approximation from $\phi : K \to K'$ to $\psi : L \to L'$ and $|\frac{t}{1-t} - \frac{q}{1-q}| < \delta$. Let $\gamma = \gamma_1 \cdots \gamma_N$ in $W_t(\phi)$ from z_0 to z_1 with each γ_i very linear, and $w_i \in W_q(\psi)$, such that $z_i(R, R')w_i$, for $i = 0, 1$.

By choice of δ_2 there is an ϵ_2-approximation (S, S'), such that for each point $z \in W_t(\phi)$ there is

$w \in W_q(\psi)$, such that $z(S,S')w$. By choice of δ_1 there is a path $\zeta = \zeta_1 \cdots \zeta_n$ in $W_q(\psi)$ from w_0 to w_1, such that each ζ_i is a piecewise linear taut path and γ_i, ζ_i are ϵ-close, for all $i = 1, \ldots, N$. ■

The next lemmas are delicate. By no means is the map $(\phi,(s,t)) \mapsto \phi_{st}$ uniformly continuous. There are nearby morphisms ϕ, ψ from a segment to a tree, such that ϕ_{tt}, ψ_{tt} are far apart for some t. See Figure 3.



Two morphisms from a segment to a tree
Figure 3

4.5 Lemma. *Let $\phi : K \to K'$ be a morphism with K, K' finite simplicial trees. For all $\epsilon > 0$ there is a $\delta > 0$, such that if (R,R') is a δ-approximation from $\phi : K \to K'$ to $\psi : L \to L'$ and $|\frac{t}{1-t} - \frac{q}{1-q}| < \delta$, then*

$$d([z_0]_t, [z_1]_t) < \epsilon,$$

for all $z_i \in W_t(\phi)$, $w_i \in W_q(\psi)$, such that $z_i(R,R')w_i$, for $i = 0,1$; and $[w_0]_q = [w_1]_q$.

Proof. Let ϵ be given. Let N be as in Lemma 4.1. Take $\epsilon_1 = \epsilon/N$. Let ϵ_1, δ_1 be as in Lemma 4.2. Take $\delta = \delta_1$. Suppose (R,R') is a δ-approximation from $\phi : K \to K'$ to $\psi : L \to L'$ and $|\frac{t}{1-t} - \frac{q}{1-q}| < \delta$. Let $z_i \in W_t(\phi)$, $w_i \in W_q(\psi)$, such that $z_i(R,R')w_i$, for $i = 0,1$; and $[w_0]_q = [w_1]_q$.

Let $z_i = (b_i, b_i')$, $w_i = (a_i, a_i')$, $i = 0,1$. By the definition of $W_q(\psi)$, we have $a_0' = a_1'$ and $d(\psi(a), a_0') \le \frac{q}{1-q}$, for all a in the segment $[a_0, a_1]$. It follows that for all b in $[b_0, b_1]$, $d(\phi(b), b_0') < \frac{t}{1-t} + \epsilon_1$. Therefore $d([z_0]_t, [z_1]_t) < \epsilon$. ■

4.6 Lemma. *Let $\phi : K \to K'$ be a morphism with K, K' compact. For all $\epsilon > 0$ there is a $\delta > 0$, such that if (R,R') is a δ-approximation from $\phi : K \to K'$ to $\psi : L \to L'$ and $|\frac{t}{1-t} - \frac{q}{1-q}| < \delta$, then*

$$|d([z_0]_t, [z_1]_t) - d([w_0]_q, [w_1]_q)| < \epsilon,$$

for all $z_i \in W_t(\phi)$, $w_i \in W_q(\psi)$, such that $z_i(R,R')w_i$, for $i = 0,1$.

Proof. Let ϵ be given. First consider the case T is a finite simplicial tree. Let N be as in Lemma 4.1. Set $\epsilon_1 = \epsilon_2 = \epsilon/(10N^2)$. Take the pairs ϵ_1, δ_1 and ϵ_2, δ_2 as in Lemmas 4.4 and 4.5, respectively.

Take $\delta = \min\{\delta_1, \delta_2\}$. Suppose (R, R') is a δ-approximation from $\phi : K \to K'$ to $\psi : L \to L'$ and $|\frac{t}{1-t} - \frac{q}{1-q}| < \delta$. Let $z_i \in W_t(\phi)$, $w_i \in W_q(\psi)$, such that $z_i(R, R')w_i$, for $i = 0, 1$.

By the choice of N there a taut path $\gamma = \gamma_1 \cdots \gamma_N$ in $W_t(\phi)$ from z_0 to z_1, with each γ_i very linear. By choice of δ_1 there is there is $\zeta = \zeta_1 \cdots \zeta_N$ in $W_q(\psi)$ from w_0 to w_1 with each ζ_i piecewise linear taut, such that γ_i, ζ_i are ϵ_1-close, for all $i = 1, \ldots, N$.

We first argue that $d([w_0]_q, [w_1]_q)$ is not much larger than $d([z_0]_t, [z_1]_t)$. Clearly

$$\begin{aligned} d([w_0]_q, [w_1]_q) &\leq \sum length(\zeta_i) \\ &< \sum(length(\gamma_i) + \epsilon_1) \\ &< (\sum length(\gamma_i)) + N\epsilon_1 \\ &\leq d([z_0]_t, [z_1]_t) + N\epsilon_1 \\ &< d([z_0]_t, [z_1]_t) + \epsilon. \end{aligned}$$

We now argue by contradiction that $d([w_0]_q, [w_1]_q)$ is not much smaller than $d([z_0]_t, [z_1]_t)$. Suppose $d([w_0]_q, [w_1]_q) \leq d([z_0]_t, [z_1]_t) - \epsilon$. As above

$$\begin{aligned} d([z_0]_t, [z_1]_t) &= \sum length(\gamma_i) \\ &< \sum length(\zeta_i) + N\epsilon_1. \end{aligned}$$

It follows that $d([w_0]_q, [w_1]_q) < (\sum length(\zeta_i)) - \frac{\epsilon}{2}$. So ζ is not taut. Since ζ folds at at most N points, there are $0 \leq a_0 < a_1 \leq 1$, such that the length along the path $\zeta|[a_0, a_1]$ is greater than $\frac{\epsilon}{4N}$, but $[\zeta(a_0)]_q = [\zeta(a_1)]_q$.

Therefore $d([\gamma(a_0)]_t, [\gamma(a_1)]_t) > \frac{\epsilon}{4N} - N\epsilon_1$. And the choice of δ_2 implies that $d([\gamma(a_0)]_t, [\gamma(a_1)]_t) < \epsilon_2$. This is a contradiction.

Finally, if T is not a finite simplicial tree, then T may be approximated close enough by a finite simplicial tree, so that the same δ works. ∎

Let $\phi : K \to K', \psi : L \to L'$ be morphisms and (R, R') an ϵ-approximation from $\phi : K \to K'$ to $\psi : L \to L'$. For any $0 \leq t, q \leq 1$ define a relation $[R, R'] = [R, R']_{tq}$ from K_t to L_q by $[z][R, R'][w]$, whenever $z(R, R')w$.

4.7 Lemma. *Let $\phi : K \to K'$ be a morphism with K, K' compact. For all $\epsilon > 0$ there is $\delta > 0$, such that if (R, R') is a δ-approximation from $\phi : K \to K'$ to $\psi : L \to L'$ and $|\frac{t}{1-t} - \frac{q}{1-q}| < \delta$, then $[R, R']_{tq}$ is an ϵ-approximation from K_t to L_q.*

Furthermore, if R and R' are P-equivariant, then $[R, R']_{tq}$ is P-equivariant.

Proof. Let $\phi : K \to K'$ be a morphism with K compact and $\epsilon > 0$ be given. Choose $\delta > 0$ by Lemma 4.6. We will show that δ has the above stated properties.

Suppose (R, R') is a δ-approximation from $\phi : K \to K'$ to $\psi : L \to L'$ and $|\frac{t}{1-t} - \frac{q}{1-q}| < \delta$. The choice of δ implies $|d([z_0]_t, [z_1]_t) - d([w_0]_q, [w_1]_q)| < \epsilon$, whenever $[z_0]_t[R, R']_{tq}[w_0]_q$ and $[z_1]_t[R, R']_{tq}[w_1]_q$. So $[R, R']_{tq}$ is an ϵ-approximation.

It follows from the definition that if R and R' are P-equivariant, then $[R, R']_{tq}$ is P-equivariant. ∎

Let $\phi : T \to T'$. If K, K' are sub-R-trees of T, T', respectively and $\phi(K) \subset K'$, then $W_t(\phi : K \to K') = W_t(\phi) \cap (K \times K')$. Moreover, $\ell \cap (K \times K')$ is connected for each element $\ell \in \mathcal{F}_t$. So define K_t as $W_t(\phi : K \to K')/(\mathcal{F}_t|K \times K')$. The induced map $K_t \to T_t$ is a monomorphism.

4.8 Theorem. *Let G be a group; $\mathcal{X}$ the space of all actions of G on* R*-trees; and $\mathcal{C} = \mathcal{C}(\mathcal{X})$ the space of all morphisms between elements of $\mathcal{X}$. Then the function $\mathcal{C} \times \nabla \to \mathcal{C}$ defined as $(\phi, (s,t)) \mapsto \phi_{st}$ is continuous.*

Proof. Let $\phi : T \to T'$ and $(s,t) \in \nabla$. Let U be an open neighborhood of ϕ_{st}. Without loss of generality suppose $U = U(\phi_{st}, K_s \times K_t, P, \epsilon)$, where K, K' are sub-R-trees, P is a finite subset of G, and $\epsilon > 0$. Take $\delta > 0$ as in Lemma 4.7 and let $V = U(\phi, K \times K', P, \delta)$.

Let $\psi : Y \to Y'$ and suppose $\psi \in V$ and $|p - s| < \delta, |q - t| < \delta$. Then there are sub-R-trees L, L' and a δ-approximation (R, R') from $\psi : K \to K'$ to $\psi : L \to L'$.

We will show that $([R,R']_{sp}, [R,R']_{tq})$ is a closed ϵ-approximation from $\psi_{st} : K_s \to K_t$ to $\psi_{pq} : L_p \to L_q$. Since (R, R') is closed, both $[R,R']_{sp}$ and $[R,R']_{tq}$ are closed. The choice of δ implies $[R,R']_{tq}$ is an ϵ-approximation from K_t to L_q and $[R,R']_{sp}$ is an ϵ-approximation from K_s to L_p. Secondly, if $[z]_s[R,R']_{sq}[w]_q$, then clearly $[z]_t[R,R']_{tq}[w]_q$.

Also by Lemma 4.7 $([R,R']_{sp}, [R,R']_{tq})$ is P-equivariant. Thus $\psi_{pq} \in U$. ∎

Though we will not use the following corollary, it is an interesting consequence.

4.9 Corollary. *Let G be a group, $\mathcal{X}$ be the space of non-trivial actions of G on* R*-trees and $\mathcal{C} = \mathcal{C}(\mathcal{X})$ the space of all morphisms between elements of $\mathcal{X}$. Then $\mathcal{C}$ is homotopy equivalent to $\mathcal{X}$.*

Proof. By Theorem 4.8 there is a map $\mathcal{C} \times \nabla \to \mathcal{C}$. Define $H : \mathcal{C} \times [0,1] \to \mathcal{C}$, by $H_t(\phi) = \phi_{t1}$. This is continuous by Proposition 2.3. Clearly $H_0(\phi) = \phi_{01} = \phi$ and $H_1(\phi) = \phi_{11} = Id_{\mathcal{R}(\phi)}$. Thus H is a strong deformation retract. By Proposition 2.5 the image of H_1 is homeomorphic to $\mathcal{X}$. ∎

5. Base Points.

The point of this section is to show that one may continuously choose a base point over a certain subspace of the space of actions of a group on R-trees.

Let G be a finitely generated group. Its is easy to see that the space of all non-trivial actions of G on R-trees strong deformation retracts to the space of all non-trivial, semi-simple actions of G on R-trees. Thus we restrict attention to this latter space.

We now generalize the notion of length function and characteristic set. Let S be be a finite subset of G. Define $\ell(S) = \ell_T(S) = \min_{x \in T} \max_{g \in S} d(x, g(x))$. The *characteristic set* of S is $T_S = \{x \in T \mid \max_{g \in S} d(x, g(x)) = \ell(S)\}$.

5.1 Lemma. *Let $G \times T \to T$ be an action on an* R*-tree and S a finite subset of G. Then T_S is contained in the union of a finite simplicial tree and $R = \bigcap T_g$, where g ranges over all elements of S satisfying $\ell(g) = \ell(S)$.*

In particular, if R is a finite simplicial tree, then T_S is a finite simplicial tree.

Proof. Arbitrarily choose a point $x \in T$. For each $g \in S$ there is a unique shortest segment from x

to T_g. Let X be the union of these segments. Clearly X is a finite simplicial tree. It suffices to show $T_S \subseteq X \cup R$.

Given y not in $X \cup R$, if z is the closest point in X to y, then $\sum_{g \in S} d(z, g(z)) < \sum_{g \in S} d(y, g(y))$. This completes the proof. ■

Let G be a finitely generated group and endow G with the metric $d(g,h) = 1$, $g \neq h$ and let G act on itself by multiplication. Let $\mathcal{X}$ be a space of non-trivial, semi-simple actions of G on $\mathbb{R}$-trees and $\mathcal{C} = \mathcal{C}(G, \mathcal{X})$ be the space of all equivariant maps of G to elements of $\mathcal{X}$. We will now define a function $b : \mathcal{X} \to \mathcal{C}$, such that $\mathcal{R}(B(T)) = T$, for all $T \in \mathcal{X}$.

Fix a finite generating set S for G. Let $G \times T \to T$ be an action on an $\mathbb{R}$-tree.

If it is irreducible or it is reducible and dihedral, then R is finite. By Lemma 5.1 T_S is a finite $\mathbb{R}$-tree. Let D be the diameter of T_S. Choose x_* to be the unique point such that T_S is contained in a closed ball of radius $D/2$ centered at x_*. If it is reducible and a shift, then choose $x_* \in T$ arbitrarily.

Now define $b(T) : G \to T$ by $g \mapsto g(x_*)$. We now check that this choice varies continuously. Recall the useful identity $d(x, g(x)) = 2d(x, T_g) + \ell(g)$.

5.2 Proposition. *Let G be a finitely generated group; $\mathcal{X}$ the space of non-trivial semi-simple actions of G on $\mathbb{R}$-trees; and $\mathcal{C} = \mathcal{C}(G, \mathcal{X})$ the space of all equivariant maps of G to elements of $\mathcal{X}$. The associated base point function $b : \mathcal{X} \to \mathcal{C}$ is continuous.*

Proof. Let $T \in \mathcal{X}$ and let U be a neighborhood of $b(T)$. The proof breaks into cases.

Case $\ell(S) = 0$. This case is vacuous – for $\ell(S) = 0$ and G finitely generated imply the action is trivial [Cu-Mo], [A-B].

Case $\ell(S) > \max_{g \in S} \ell(g)$. In this case T_S is a single point and there are $g_0, g_1 \in S$, such that $T_S = \{x | d(x, g_i(x) = \ell(S), i = 0, 1\}$. Let $T_S = \{x_*\}$. Let x_i be the point on T_{g_i} nearest to x_*. In particular $x_* \in [x_0, x_1]$ and $d(x_0, x_1) = d(x_0, x_*) + d(x_*, x_1)$.

Without loss of generality suppose $U = U(b(T), P \times K, P, \epsilon)$, where $S \subseteq P$ and $P(x_*) \subseteq K$. Take $V = U(T, K, P, \delta)$, where $\delta = \frac{1}{8} \min\{\epsilon, d(x_*, T_{g_0}), d(x_*, T_{g_1}\}$. Suppose $Y \in V$. By definition there is a P-equivariant, closed δ-approximation R from K to L for some $L \subset Y$.

Take $y_*, y_0, y_1 \in L$, such that $x_* R y_*$ and $x_i R y_i$, $i = 0, 1$. By definition $|d(y_0, y_1) - d(y_0, y_*) - d(y_*, y_1)| < 2\delta$. Also $|d(x_*, g_i(x_*)) - d(y_*, g_i(y_*))| < \delta$ and $|d(x_i, g_i(x_i)) - d(y_i, g_i(y_i))| < \delta$. It now follows that

$$\begin{aligned} |d(y_*, Y_{g_i}) - d(y_*, y_i) - d(y_i, Y_{g_i})| &= |\tfrac{1}{2}(d(y_*, g_i(y_*)) - \ell_Y(g_i)) - d(y_*, y_i) - \tfrac{1}{2}(d(y_i, g_i(y_i)) - \ell_Y(g_i))| \\ &= |\tfrac{1}{2}(d(y_*, g_i(y_*)) - d(y_i, g_i(y_i))) - d(y_*, y_i))| \\ &< 2\delta. \end{aligned}$$

Using the above inequalities one may show that $|\ell_T(S) - \ell_Y(S)| < 2\delta$. Therefore Y_S is a point and $d(y_*, Y_S) < 2\delta$. Now observe that $(Id, R_{2\delta})$ is a P-equivariant, closed ϵ-approximation from $b(T) : P \to K$ to $b(Y) : P \to Y$. So $b(Y) \in U$.

Case $\ell(S) = \max_{g \in S} \ell(g) > 0$ and T is T_S is finite. Let $g \in S$, such that $\ell(G) = \ell(G)$. So there are points x_0, x_1, such that $T_S = [x_0, x_1]$. If $x_0 \neq x_1$, choose orientation so that g translates from x_0 to x_1. Consider the segment $[g^{-1}(x_0), g(x_1)]$. Since $g^{-1}(x_0), g(x_1)$ do not lie in T_S, there exists

$g_0, g_1 \in S$ (possibly equal), such that $d(g^{-1}(x_0), g_0(g^{-1}(x_0))) = \ell(S) + 2\ell(g)$ and $d(g(x_1), g_0(g(x_1))) = \ell(S) + 2\ell(g)$.

Without loss of generality suppose $U = U(b(T), P \times K, P, \epsilon)$, where $S, S^{-1} \subseteq P$ and $P([g^{-1}(x_0), g(x_1)]) \subseteq K$. Take $V = U(T, K, P, \delta)$, where $\delta = \frac{1}{8}\min\{\epsilon, \ell(g)\}$. Suppose $Y \in V$. By definition there is a P-equivariant, closed δ-approximation R from K to L for some $L \subset Y$.

Take $y_*, y_0, y_1 \in L$, such that $x_* R y_*$ and $x_i R y_i$, $i = 0, 1$. Arguing as above using all the points $y_*, y_0, y_1, g^{-1}(y_0), g(y_1)$ one may show that the Hausdorff distance between $[y_0, y_1]$ and Y_S is less than 2δ. As above it follows that $d(y_*, b(Y)(1)) < 2\delta$. Now the proof is completed as above.

Case $\ell(S) = \max_{g \in S} \ell(g) > 0$ and T is T_S is infinite. In this case T is reducible and a shift. This is proved as the previous case. ∎

Proposition 5.2 is definitely false if we let $\mathcal{X}$ be the space of all actions. In particular, there is no continuous choice of base point for a trivial, non-minimal action with a non-compact fixed point set nor a non-trivial action with exactly one fixed end.

6. Free Groups.

Let V_n be a wedge of n circles. Choose a CW-complex structure on V_n with exactly one vertex. Let $\tilde{V}_n$ be the universal covering. Let F_n be the free group of rank n. Identify F_n with the fundamental group of V_n based at the vertex. So F_n acts on $\tilde{V}_n$.

A *lamination* $\mathcal{L}$ on V_n is a non-empty, closed, 0-dimensional subset of the 1-skeleton. A *transverse measure* on $\mathcal{L}$ is a function μ from the set of arcs transverse to $\mathcal{L}$, such that (i) $\mu(\gamma+\gamma') = \mu(\gamma)+\mu(\gamma')$; and (ii) $\mu(\gamma) = \mu(\gamma')$, whenever γ, γ' have the same image.

A measured lamination $(\mathcal{L}, \mu)$ in V_n has a lift $(\tilde{\mathcal{L}}, \tilde{\mu})$ in $\tilde{V}_n$. Say the action $F_n \times T \to T$ is *dual* to the measured lamination $(\mathcal{L}, \mu)$ if there is an equivariant, locally constant map $p : \tilde{V}_n - \tilde{\mathcal{L}} \to T$, such that $\tilde{\mu}(\gamma) = d(p(\gamma(0)), p(\gamma(1)))$, for every transverse arc $\gamma : [0,1] \to \tilde{V}_n$ meeting each leaf of $\mathcal{L}$ at most once. Clearly every measured lamination on V_n is dual to an action on an R-tree and the space of measured laminations on V_n is homeomorphic to a closed $(n-1)$-simplex $\times \mathbf{R}^+$.

Say a map $\tilde{V}_n \to T$ is *transverse* if it is linear on every 1-simplex.

6.1 Lemma. *Let $\mathcal{X}$ be a space of non-trivial semi-simple actions of F_n on* R*-trees and let $\mathcal{C} = \mathcal{C}(\tilde{V}_n, \mathcal{X})$ be the space of all equivariant transverse maps of $\tilde{V}_n$ to elements of $\mathcal{X}$. Then there is a map $B : \mathcal{X} \to \mathcal{C}$, such that $\mathcal{R}(B(T)) = T$, for all $T \in \mathcal{X}$.*

Proof. Choose an arbitrary finite generating set S for F_n. By Proposition 5.2 the associated function $b : \mathcal{X} \to \mathcal{C}(F_n, \mathcal{X})$ is continuous.

Let $F_n \times T \to T$ be a non-trivial, semi-simple action. We start by defining $\tilde{V}_n \to T$ on the 0-skeleton. Identify the action of F_n on the 0-skeleton of $\tilde{V}_n$ with the action of F_n on itself by multiplication. Define $\tilde{V}_n \to T$ on the 0-skeleton by $b(T)$.

Now extend to the 1-skeleton linearly. This defines $B(T) : \tilde{V}_n \to T$. Clearly this is transverse. Its easy to see that this varies continuously with $F_n \times T \to T$. ∎

Let $\mathcal{G} \subseteq \mathcal{X}$ be the space of non-trivial actions dual to measured laminations on V_n.

6.2 Lemma. *Let $\mathcal{X}$ be the space of all non-trivial semi-simple actions of F_n on* R*-trees and $\mathcal{C} = \mathcal{C}(\mathcal{X})$ the space of all morphisms between elements of $\mathcal{X}$. Then there is a map $r : \mathcal{X} \to \mathcal{C}$, such that*

(i) $\mathcal{D}(r(T)) \in \mathcal{G}$ and $\mathcal{R}(r(T)) = T$, for all $T \in \mathcal{X}$ and

(ii) $r(T) = Id_T$, for all $T \in \mathcal{G}$.

Proof. Let $B : \mathcal{X} \to \mathcal{C}(\tilde{V}_n, \mathcal{X})$ be the map of Lemma 6.1. For each action $F_n \times T \to T$ and transverse map $B(T) : \tilde{V}_n \to T$ we will define a measured lamination on V_n.

Choose any invariant, countable, dense, 0-dimensional subset D of T, such that it contains every vertex of T and contains the image of every vertex of $\tilde{V}_n$. We want the pre-image of the complement of D to be a lamination.

Now equivariantly change the map on the interior of each 1-simplex to a new monotone map, such that the pre-image of D is an open dense subset of the simplex. This determines a measured lamination on $\tilde{V}_n$ where the measure of each transverse arc is the length of its image in T. Let $(\mathcal{L}, \mu)$ be the induced measured lamination on V_n.

Clearly these measured laminations vary continuously with $F_n \times T \to T$. If R is the R-tree dual to $(\mathcal{L}, \mu)$, then define $r(T)$ to be the morphism $R \to T$ which factors $\tilde{V}_n \to T$. Thus $r : \mathcal{X} \to \mathcal{C}$ is a map with the desired properties. ∎

6.3 Theorem. *Let $\mathcal{X}$ be the space of all non-trivial, semi-simple actions of F_n on* R*-trees. Then $\mathcal{X}$ is contractible.*

Proof. Let $r : \mathcal{X} \to \mathcal{C}$, $(\phi, (s,t)) \mapsto \phi_{st}$, and $\mathcal{R} : \mathcal{C} \to \mathcal{X}$ be the maps of Lemma 6.2, Theorem 4.4, and Proposition 2.4, respectively.

By Proposition 3.8 $\phi_{st} \in \mathcal{C}$. Define $\tilde{H} : \mathcal{X} \times [0,1] \to \mathcal{X}$ by $\tilde{H}_{(1-t)}(T) = \mathcal{R}(\phi_{0t})$, where $\phi = r(T)$. Then $\tilde{H}$ is a strong deformation retract of $\mathcal{X}$ to $\mathcal{G}$ which is contractible. ∎

Let G be a group and let $\mathcal{X}$ be the space of minimal semi-simple actions of G on R-trees. Let $\ell : \mathcal{X} \to \mathcal{LF}(\mathcal{G})$ be the map which assigns to each action its length function. By [Cu-Mo] it is bijective. Paulin [Pau2] showed that this function restricted to the space of irreducible actions is an embedding. He proves this by working only with the definitions of the two topologies. His techniques actually work to show that ℓ is a homeomorphism.

The multiplicative group R^+ acts on $\mathcal{X} - 0$ by homothety. So the induced map $[\ell] : (\mathcal{X} - 0)/\mathrm{R}^+ \to \mathcal{PLF}(G)$ is also a homeomorphism.

We will need some notation. Let G be a group and S a finite subset of G. Let $\mathcal{X}$ be the space of actions of G on R-trees. Define $||\cdot|| : \mathcal{X} \to \mathrm{R}$ by $||T|| = \ell_T(S)$. It satisfies $||\alpha T|| = \alpha ||T||$ and it is continuous. It follows from above that $[\ell]|\{T \mid T$ is non-trivial, minimal semi-simple and $||T|| = 1\}$ is a homeomorphism.

Recall that our deformation may not leave invariant the sub-space of minimal actions. This problem has a few simple solutions. We solve it by working with length functions in the next theorem.

6.4 Theorem. *The space $\mathcal{PLF}(F_n)$ is contractible.*

Proof. Let $\mathcal{X}$ be the space of non-trivial semi-simple actions of F_n on R-trees. Let $\tilde{H}$ be the strong deformation retract in the proof of Theorem 6.3. From the above discussion there is a stong deformation retract $H : \mathcal{PLF}(F_n) \times [0,1] \to \mathcal{PLF}(F_n)$ covered by $\tilde{H}|\{T \in \mathcal{X} \mid ||T|| = 1\}$. As above the image of H_1 is contractible. ∎

The space of projective *small* length functions $\mathcal{SLF}(F_n) \subseteq \mathcal{PLF}(F_n)$ is the set of length functions corresponding to actions where the stablizer of any edge does not contain a free group of rank two.

Let $T \to T'$ be a morphism. If T' is small (free) then T is small (free). The next two results are obvious.

6.5 Theorem. *The space $\mathcal{SLF}(F_n)$ is contractible.* ∎

6.6 Theorem. *The space of free actions of F_n on* R*-trees is contractible.* ∎

Define $\mathcal{CV}(F_n) \subseteq \mathcal{PLF}(F_n)$ to the image of length functions of actions of F_n on simplicial trees. Marc Culler and Karen Vogtmann [Cu-Vo1] constructed this space to study the outer automorphism group of F_n. They also showed it was contractible using combinatorial methods. We will show that it also follows from Theorem 6.4.

6.7 Theorem. [Culler-Vogtmann] *The space $\mathcal{CV}(F_n)$ is contractible.*

Proof. It suffices to see that the deformation H of Theorem 6.4 leaves $\mathcal{CV}(F_n)$ invariant. Let $\phi_{s1} : T_s \to T_1$. If T_1 is simplicial and the action on T_1 is free, then the action on T_1 is properly discontinuous. Thus the action on T_s is properly discontinuous. It follows that T_s has an invariant simplicial sub-tree and the action on T_s is free. ∎

Let $\bar{\mathcal{CV}}(F_n)$ be the closure of $\mathcal{CV}(F_n)$. M. Steiner [St2] showed that it was contractible.

6.8 Theorem. [Steiner] *The space $\bar{\mathcal{CV}}(F_n)$ is contractible.*

Proof. Again it suffices to see that the deformation H of Theorem 6.4 leaves $\bar{\mathcal{CV}}(F_n)$ invariant. This follows from the proof of Theorem 6.7. ∎

7. Surface Groups.

Let F be a closed hyperbolic surface. Let $\mathbf{H}^2 \to F$ be the universal covering and let $\pi_1 F$ act on $\mathbf{H}^2$ by covering transformations. A *lamination* $\mathcal{L}$ is a non-empty, closed subset of F, such that each point in $\mathcal{L}$ has a neighborhood U and a homeomorphism $(U, U \cap \mathcal{L}) \to ([0,1] \times [0,1], [[0,1] \times Z)$, for some 0-dimensional subset Z of $[0,1]$. such that each path component is a simple geodesic. A *geodesic lamination* is a lamination such that each path component is a geodesic. A *transverse measure* μ on a lamination $\mathcal{L}$ is a function from the set of paths transverse to $\mathcal{L}$ to $[0, +\infty)$, such that (i) $\mu(\gamma + \gamma') = \mu(\gamma) + \mu(\gamma')$; and (ii) $\mu(\gamma) = \mu(\gamma')$, whenever γ, γ' are homotopic through a 1-parameter family of transverse paths.

A measured lamination $(\mathcal{L}, \mu)$ in F has a lift $(\tilde{\mathcal{L}}, \tilde{\mu})$ in $\mathbf{H}^2$. Say the action $\pi_1 F \times T \to T$ is *dual* to a measured lamination $(\mathcal{L}, \mu)$ if there is an equivariant, locally constant map $p : \mathbf{H}^2 - \tilde{\mathcal{L}} \to T$, such that $\tilde{\mu}(\gamma) = d(p(\gamma(0)), p(\gamma(1)))$, for every transverse path $\gamma : [0,1] \to \mathbf{H}^2$ meeting each leaf of $\mathcal{L}$ at most once. Every measured geodesic lamination is dual to an action on an $\mathbf{R}$-tree [Hat], [Mo-Sh4]. William P. Thurston [Th] proved that the space of measured geodesic laminations is homeomorphic to $S^n \times \mathbf{R}^+$, where $n = -3\chi(F)$.

To prove the next two lemmas it will be convenient to fix an *ideal triangulation* of F, i.e. a CW-complex structure with exactly one 0-simplex and with every 2-simplex meeting exactly three 1-simplicies. Lift this ideal triangulation to a triangulation of $\mathbf{H}^2$.

Say a map $\mathbf{H}^2 \to T$ is *transverse* if it is an embedding on every 1-simplex.

7.1 Lemma. *Let $\mathcal{X}$ be a space of non-trivial, semi-simple actions of $\pi_1 F$ on $\mathbf{R}$-trees and let $\mathcal{C} = \mathcal{C}(\mathbf{H}^2, \mathcal{X})$ be the space of all equivariant transverse maps of $\mathbf{H}^2$ to elements of $\mathcal{X}$. Then there is a map $B : \mathcal{X} \to \mathcal{C}$, such that $\mathcal{R}(B(T)) = T$, for all $T \in \mathcal{X}$.*

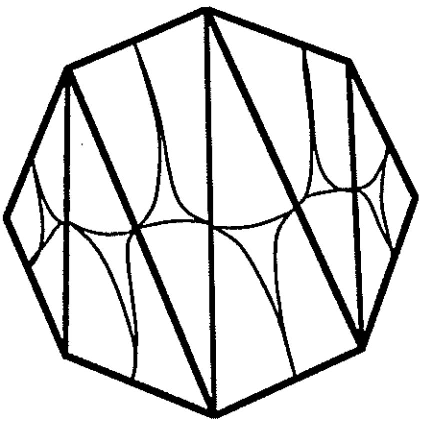
Proof. Take any generating set for $\pi_1 F$. By Proposition 5.2 there is a base point map $b : \mathcal{X} \to \mathcal{C}(\pi_1 F, \mathcal{X})$.

Let $\pi_1 F \times T \to T$ be a non-trivial, semi-simple action. We start by defining $\mathbf{H}^2 \to T$ on the 0-skeleton. Identify the action of $\pi_1 F$ on the 0-skeleton of $\mathbf{H}^2$ with the action of F_n on itself by multiplication. Now define $\mathbf{H}^2 \to T$ on the 0-skeleton by $b(T)$.

Extend to the 1-skeleton linearly. Finally, extend to the 2-skeleton as follows. Subdivide each 2-simplex by coning to the centroid. Map each 2-simplex to T by sending the centroid to the centroid of the image of its verticies and coning the rest. (The *centroid* of three points a, b, c in an $\mathbf{R}$-tree, is the unique point x which minimizes $d(a,x) + d(b,x) + d(c,x)$.) This defines $B(T) : \mathbf{H}^2 \to T$. Clearly this is transverse and continuous.

■

Laminations on a surface are more complicated than laminations on a 1-complex. We need the following tool to study laminations on a surface [Th], [Pe-Ha]. A *smooth* graph is an embedded graph $\tau \subseteq F$, such that for each point $p \in \tau$, there is a smooth, open arc in τ through p and any two such arcs are tangent at p. A *train track* is a smooth graph τ, such that for each component C of $F - \tau$, the double of C along its smooth frontier has negative Euler characteristic. A train track τ *carries* a lamination $\mathcal{L}$ if there is a homotopy equivalence $f : F \to F$, such that $f|\alpha : \alpha \to \tau$ is a smooth immersion for each smooth arc $\alpha \subseteq \mathcal{L}$. Every geodesic lamination is carried by a (non-unique) train track.



A smooth graph in a surface
Figure 4

A. Hatcher [Ha] and Morgan and Otal [Mo-Ot] proved that given an action $\pi_1 F \times T \to T$, there is an action $\pi_1 F \times R \to R$ and morphism $R \to T$, such that $\pi_1 F \times R \to R$ is dual to a measured geodesic lamination. It is easy to see that the construction of Hatcher varies continuously with T. We will sketch his proof in the next lemma.

Let $\mathcal{G}$ be the space of non-trivial actions of $\pi_1 F$ dual to measured geodesic laminations on F.

7.2 Lemma. *Let F be a closed hyperbolic surface; $\mathcal{X}$ be the space of all non-trivial, semi-simple actions of $\pi_1 F$ on* R*-trees; and $\mathcal{C} = \mathcal{C}(\mathcal{X})$ the space of all morphisms between elements of $\mathcal{X}$. Then there is a map $r : \mathcal{X} \to \mathcal{C}$, such that*

(i) $\mathcal{D}(r(T)) \in \mathcal{G}$ and $\mathcal{R}(r(T)) = T$, for all $T \in \mathcal{X}$; and

(ii) $r(T) = Id_T$ for all $T \in \mathcal{G}$.

Proof. Let $B : \mathcal{X} \to \mathcal{C}(\mathrm{H}^2, \mathcal{X})$ be the map of Lemma 7.1. For each action $\pi_1 F \times \mathrm{H}^2 \to \mathrm{H}^2$ and transverse map $\mathrm{H}^2 \to T$ we will define a lamination on F.

Choose any invariant, countable, dense, 0-dimensional subset D of T, such that it contains every vertex of T and contains the image of every vertex of H^2.

Now equivariantly change the map on the interior of the 1-skeleton to a new monotone map, such that the pre-image of D is an open dense subset of of the 1-skeleton. Do the same for the 2-simplicies. This determines a measured lamination $(\mathcal{L}, \mu)$ on F, such that the measure of each 1-simplex in F is the length of the image of any lift in T.

This lamination is carried by a smooth graph τ, where each 2-simplex in the triangulation of F contains exactly 3 edges of τ. See Figure 4. There is one complementary region which prevents τ from being a train track. We will argue that some sub-graph is a train track which also carries the essential leaves of the lamination. The measure determines weights on the graph edges. Suppose all the weights on τ are non-zero. Since τ bounds a disk, there are trivial leaves in the lamination. Remove the trivial leaves. Now at least one of the edges of τ bounding the disk has weight zero, so we may throw out that edge. Also by construction of the transverse map no lift of a leaf of the lamination to $\mathbf{H}^2$ meets a lift of an edge more than once. So at least one more edge has weight zero and we may throw out that edge. Finally we may isotop the lamination and remove one more edge. Thus the lamination is carried by a train track. It follows that there is a dual action $\pi_1 F \times R \to T$ and a morphism $R \to T$. Finally, we may replace the measured lamination by a measured geodesic lamination.

The weights on the train track vary continuously. By [Th] these measured laminations vary continuously with $\pi_1 F \times \mathbf{H}^2 \to \mathbf{H}^2$. Define $r(T)$ to be the morphism $R \to T$. This completes the proof. ∎

The last two theorems follow as in §6.

7.3 Theorem. *Let F be a closed hyperbolic surface and $\mathcal{X}$ be the space of all non-trivial semi-simple actions of $\pi_1 F$ on* R*-trees. Then $\mathcal{X}$ strong deformation retracts to a sphere of dimension $-3\chi(F)$.* ∎

7.4 Theorem. *Let F be a closed hyperbolic surface. Then $\mathcal{PLF}(\pi_1 F)$ strong deformation retracts to a sphere of dimension $-3\chi(F)$.* ∎

All the theorems of this section have an analog for a compact surface with boundary F. Since the fundamental group of F is a free group, one should take the geometry of the surface into consideration. A natural class of actions is the space of actions of $\pi_1 F$ on R-trees, such that $\pi_1 E$ fixes a point for each component E of ∂F.

The methods of these last two sections generalize. The key ingredient is the space $\mathcal{G}$ in Lemmas 6.2 and 7.2. For example let $G = G_1 * G_2$ and take $\mathcal{G}$ to be the join of $\mathcal{PLF}(G_1)$ with $\mathcal{PLF}(G_2)$. Then $\mathcal{PLF}(G)$ strong deformation retracts to $\mathcal{G}$.

References.

[A-B] R. Alperin and H. Bass, "Length Functions of Group Actions on Λ-Trees," in: S. M. Gersten et al., eds., *Combinatorial Group Theory and Topology*, Annals of Math Studies #111, Princeton University Press, Princeton, (1987) 265-378.

[Cul-Mor] M. Culler and J. W. Morgan, "Group Actions on R-Trees," *Proc. Lond. Math. Soc.*, **55**, (1987) 571-604.

[Cu-Vo1] M. Culler and K. Vogtmann, "Moduli of Graphs and Outer Automorphisms of Free Groups," *Invent. Math.*, 84 (1986) 91-119.

[Cu-Vo2] M. Culler and K. Vogtmann, "The Boundary of Outer Space in Rank Two," (preprint).

[Fu] K. Fukaya, "A Boundary of the the Set of the Riemannian Manifolds with Bounded Curvature and Diameters," *J. Difff. Geo.*, 28 (1988) 1-21.

[G-S] H. Gillet and P. B. Shalen, "Dendrology of Groups in low Q-ranks," *J. Diff. Geo.* (to appear).

[Gro] M. Gromov, "Groups of Polynomial Growth and Expanding Maps," *Inst. Hautes Études Sci. Publ. Math.*, 53 (1981) 53-78.

[Hat] A. Hatcher, "Measured Lamination Spaces for Surfaces from the Topological Viewpoint," *Top. Appl.*, 30, (1988), 63-88.

[Mo-Ot] J. W. Morgan and J-P. Otal, "Relative Growth Rates of Closed Geodesics on a Surface under varying Hyperbolic Structures," *Comm. Math.* (to appear).

[Mo-Sh1] J. W. Morgan and P. B. Shalen, "Degenerations of Hyperbolic Structures, I: Valuations, Trees and Surfaces," *Ann. Math.*, 120 (1984) 401-476.

[Pau1] F. Paulin, "Topologie de Gromov, équivariant, structures hyperboliques et arbres réels," *Invent. Math.*, 94 (1988) 53-80.

[Pau2] F. Paulin, "The Gromov Topology on R-Trees," *Top. Appl.*, 32 (1989) 197-221.

[P-H] R. Penner and J. Harer, "Combinatorics of Train Tracks," (pre-print).

[St1] M. Steiner, Ph.D. Thesis, Columbia University, (1988).

[St2] M. Steiner.

[Th] W. Thurston, "The Geometry and Topology of three-Manifolds," (preprint).

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