COMBINATION THEOREMS FOR ACTIONS ON TREES

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Group actions on R-trees arise naturally in combinatorial group theory [Ly], [Ch] and [A-M], in arithmetic group theory [Ti] and in the study of deformations of hyperbolic structures on groups [M-S].

The theory of Bass and Serre [Se] shows that the actions of a group G on simplicial trees are in natural one-to-one correspondence with splittings of the group G. For actions on R-trees there is no such complete structure theorem. In particular it is unknown whether every group which admits a nontrivial action on an R-tree must admit a nontrivial splitting. Thus it is both natural and useful to consider actions on R-trees by groups which admit splittings.

Suppose a group G splits as a free product with amalgamation $G_1 *_{G_0} G_2$ (or an HNN-extension $(G_1 *_{G_0}, s)$), where each factor group and amalgamating group acts on an R-tree and a certain compatiblity condition is satisfied. The main result of this paper is the construction of an action of G on an R-tree which is to the category of groups actions on R-trees as the free product with amalgamation (or HNN-extension) construction is to the category of groups (see §3). The construction in the case of the free product with amalgamation has already been done by Michael Steiner [St]. His construction is combinatorial.

Our constructions are topological and are partially motivated by the approach Scott and Wall [S-W] take to the Bass-Serre Theory. Essentialy there is an isomorphism between the category of group actions on R-trees and the category of spaces with measured foliations (§2). In order to construct an action on an R-tree we first construct a space with a measured folition. Then the action is dual to the measured foliation. This not only makes the construction of the action simple, but it also provides a way to study the action.

In §3 we construct both the free product with amalgamation and the HNN-extension in the category of group actions on trees. After constructing the free product with amalgamation, it is obvious how to construct the HNN-extension. In fact one could copy our construction for more general spittings of groups.

In §4 we prove an inverse to each of the above constructions. Thus the above constructions are completely general. More precisely, given an action on a tree by a group which splits there is a corresponding *maximal* splitting of the action. It is trivial to find a non-maximal splitting.

In §5 we give some applications of the above methods. In the first example we construct a measured foliation on a closed, genus two, orientable surface which is dual to a free action on an R-tree. By other methods it was shown by Peter B. Shalen and John W. Morgan [M-S4] that most closed surfaces have measured geodesic laminations which are dual to free actions on R-trees (in this case the difference between foliations and geodesic laminations is convenience).

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In a second example we construct a non-free action of a closed, genus two, orientable surface on a tree which is the equivariant image of the first example.

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1. The Category of Group Actions on Trees.

The following definitions are taken from [A-B], [C-M], [M-S2] and [G-S]. Let X be a metric space with metric d. For any distinct $x, y \in X$ the *segment* from x to y is $[x, y] = \{z \in X | d(x, y) = d(x, z) + d(z, y)\}$.

An **R**-*tree* is a non-empty, metric space T satisfying

(1) for all $x, y \in T, [x, y]$ is isometric to a segment in R;

(2) for all $x, y, z \in T$, if $[x, y] \cap [y, z] = \{y\}$, then $[x, z] = [x, y] \cup [y, z]$; and

(3) for all $x, y, z \in T$, there is a $w \in T$, such that $[x, y] \cap [y, z] = [y, w]$.

Notice we will let simply *tree* mean **R**-tree.

Given an **R**-tree T and $x \in T$, define $B_x = \{[x, y] \,|\, y \in T - \{x\}\}$. Define an equivalence relation by $[x, y] \sim [x, z]$ if $[x, y] \cap [x, z] = [x, w]$, for some $w \in T - \{x\}$. A *direction* at x is an equivalence class in B_x.

Let $x \in T$. If x has exactly two directions, then say x is an *edge*, otherwise x is a *vertex*. Notice that x is an edge if and only if $T - \{x\}$ has exactly two connected components.

A *morphism* from a tree T to a tree T' is a map $\phi : T \to T'$, such that for each segment $[x, y]$ there is a segment $[x, w] \subseteq [x, y]$, such that $\phi \,|\, [x, w]$ is an isometry.

A *group action on a tree* is a triple $\mathcal{T} = (G, T, \rho)$, where G is a group, T is an **R**-tree, and ρ is a homomorphism of G into the group of isometries of T which act on the left. Sometimes the action $\mathcal{T}$ will be denoted simply as $G \times T \to T$.

An action is *minimal* if there is no invariant, proper subtree. An action $G \times T \to T$ is *reducible* if either

(1) G fixes a point of T;

(2) G fixes an end of T; or

(3) G leaves invariant a set containing two ends of T .

The *category of group actions on trees* has as *objects* group actions on trees.

Given group actions on trees $\mathcal{T} = (G, T, \rho)$ and $\mathcal{T}' = (G', T', \rho')$, a *morphism* from $\mathcal{T}$ to $\mathcal{T}'$ is a pair (ω, ϕ), where $\omega : G \to G'$ is a homomorphism and $\phi : T \to T'$ is a morphism satisfying

$$\phi(\rho(g)(x)) = \rho'(\omega(g))(\phi(x)),$$

for all $g \in G, x \in T$.

2. The Category of Measured Foliations.

Some of the definitions and propositions of this section generalize [G-S].

2.1 Definition. Let X be a topological space. A *foliation* $\mathcal{F}$ of X is a decomposition of X into subsets which are path connected.

Let U be an open subset of X and T a tree. A *chart* is a map $p : U \to T$, satisfying for each $y \in T$, there is a leaf ℓ of $\mathcal{F}$, such that $p^{-1}(y)$ is a path component of $U \cap \ell$.

2.2 Definition. Let X be a topological space and $\mathcal{F}$ a foliation. A *transverse measure* μ on $\mathcal{F}$ is a collection of charts $\{p_\alpha : U_\alpha \to T_\alpha\}_\alpha$ satisfying

(1) $\{U_\alpha\}_\alpha$ is a basis for X; and

(2) if $U_\alpha \subseteq U_\beta$, then there is an isometry $T_\alpha \to T_\beta$ making the diagram

$$\begin{array}{ccc} U_\alpha & \hookrightarrow & U_\beta \\ p_\alpha \downarrow & & \downarrow p_\beta \\ T_\alpha & \rightarrow & T_\beta \end{array}$$

commute.

A *measured foliation* is a pair $(\mathcal{F}, \mu)$, where $\mathcal{F}$ is a foliation and μ is a transverse measure. The *category of measured foliations* has as *objects* pairs $\mathcal{X} = (X, (\mathcal{F}, \mu))$, where X is a topological space and $(\mathcal{F}, \mu)$ is a measured foliation. Given two objects $\mathcal{X} = (X, (\mathcal{F}, \mu))$, $\mathcal{X}' = (X', (\mathcal{F}', \mu'))$, a *morphism* from $\mathcal{X}$ to $\mathcal{X}'$ is a map $f : X \to X'$, such that for all $x \in X$ there are charts $U_\alpha \to T_\alpha$, $U_\beta \to T_\beta$ with $x \in U_\alpha$, $g(x) \in U_\beta$, and a tree morphism $T_\alpha \to T_\beta$, such that the diagram

$$\begin{array}{ccc} U_\alpha & \xrightarrow{f} & U_\beta \\ p_\alpha \downarrow & & \downarrow p_\beta \\ T_\alpha & \rightarrow & T_\beta \end{array}$$

commutes.

2.3 Definition. Let X be a path connected, locally path connected topological space with universal cover $\tilde{X}$. Let $\mathcal{X} = (X, (\mathcal{F}, \mu))$ be a measured foliation and $(\tilde{X}, (\tilde{\mathcal{F}}, \tilde{\mu}))$ the lifted measured foliation. Let $\mathcal{T} = (G, T, \rho)$ be a group action on a tree. Say that $\mathcal{X}$ is *dual* to $\mathcal{T}$ if there is an identification of $\pi_1 X$ with G and an equivariant map $p : \tilde{X} \to T$, such that

(1) for every $y \in T$, $p^{-1}(y)$ is a leaf of $\mathcal{F}$; and

(2) for every $x \in X$ there is a chart $U_\alpha \to T_\alpha$ with $x \in U_\alpha$, and monomorphism $T_\alpha \to T$ making the diagram

$$\begin{array}{ccc} U_\alpha & \hookrightarrow & \tilde{X} \\ p_\alpha \downarrow & & \downarrow p \\ T_\alpha & \to & T \end{array}$$

commute.

It is clear that if $\mathcal{X}$ is dual to both $\mathcal{T}$ and $\mathcal{T}'$, then $\mathcal{T}$ and $\mathcal{T}'$ are isomorphic. The following approximately says that the functions which takes a measured foliation to its dual is a functor.

2.4 Proposition. *Let $\mathcal{X} = (X, (\mathcal{F}, \mu))$, $\mathcal{X}' = (X', (\mathcal{F}', \mu'))$ be dual to $\mathcal{T} = (\pi_1 X, T, \rho)$, $\mathcal{T}' = (\pi_1 X', T', \rho')$, respectively. Then for every morphism $f : X \to X'$ and equivariant lift $\tilde{f} : \tilde{X} \to \tilde{X}'$ there is a unique morphism $(\omega, \phi) : \mathcal{T} \to \mathcal{T}'$, such that $\tilde{f}(g(x)) = (\omega(g))(\tilde{f}(x))$, for all $g \in G, x \in \tilde{X}$; and the diagram*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X}' \\ \downarrow p & & \downarrow p' \\ T & \xrightarrow{\phi} & T' \end{array}$$

commutes.

□

We now show that this functor is onto. Given an action $G \times T \to T$, define a dual foliation as follows. Let K be an Eilenberg-MacLane space with fundamental group G. Let $\tilde{K} \to K$ be the universal covering. The diagonal action $G \times (\tilde{K} \times T) \to \tilde{K} \times T$ is a covering action. Define a measured foliation $(\tilde{\mathcal{F}}, \tilde{\mu})$ on $\tilde{X} \times T$ whoses leaves are the $\tilde{K}$ factors and whose transverse measure is given by projection to T. Let $X = (\tilde{K} \times T)/G$ with the induced foliation $(\mathcal{F}, \mu)$. The pair $(X, (\mathcal{F}, \mu))$ is called the *realization* of $G \times T \to T$.

The following is obvious.

2.5 Proposition. *Let $\mathcal{T}$ be an action on a tree with realization $\mathcal{X} = (X, (\mathcal{F}, \mu))$. Then $\mathcal{X}$ is dual to $\mathcal{T}$.* □

Proposition 2.6 implies that the function that takes a measured foliation to its dual has an inverse. In particular, it implies that the realization is essentially unique.

2.6 Proposition. *Let $\mathcal{T} = (G, T, \rho), \mathcal{T}' = (G', T', \rho')$ have realizations $(X, (\mathcal{F}, \mu))$, $(X', (\mathcal{F}', \mu'))$, respectively. Then for any morphism $u : \mathcal{T} \to \mathcal{T}'$ there is a morphism $f : X \to X'$ and lift $\tilde{f} : \tilde{X} \to \tilde{X}'$ making the diagram*

$$\begin{array}{ccc} \tilde{X} & \to & \tilde{X}' \\ \downarrow p & & \downarrow p' \\ T & \to & T' \end{array}$$

commute.

Proof. By hypothesis $\tilde{X} = \tilde{K} \times T$, $\tilde{X}' = \tilde{K}' \times T'$ and $X = \tilde{X}/G$, $X' = \tilde{X}'/G'$. Choose basepoints in X and X' in order to make the identifications $\pi_1 K = G$ and $\pi_1 K' = G'$. Choose basepoints in $\tilde{X}$ and $\tilde{X}'$ in order to identify the fundamental groups with the covering transformations. So there is $\Omega : K \to K'$, such that $\Omega_* = \omega$. Let $\tilde{\Omega} : \tilde{K} \to \tilde{K}'$ cover Ω. Define $\tilde{f} : \tilde{X} \to \tilde{X}'$ as $\tilde{f} = \tilde{\Omega} \times \phi$. So $\tilde{f}$ covers some $f : X \to X'$. Finally, the diagram

$$\begin{array}{ccc} \tilde{X} & \to & \tilde{X}' \\ \downarrow p & & \downarrow p' \\ T & \to & T' \end{array}$$

is clearly commutative. □

For completeness we mention that a measured foliation always determines a measure on paths in the following way.

A path $\gamma : [0,1] \to T$ is *monotone* if $\gamma^{-1}(y)$ is connected for each $y \in T$. A path $\gamma : [0,1] \to X$ is *transverse* to $\mathcal{F}$ if for each $t \in [0,1]$ there is a $p_\alpha : U_\alpha \to T_\alpha$ with $\gamma(t) \in U_\alpha$, such that $p_\alpha \circ \gamma|\gamma^{-1}(U_\alpha)$ is monotone.

Given a transverse path $\gamma : [0,1] \to U_\alpha$ define the *measure* $\mu(\gamma) = d(p_\alpha(\gamma(0)), p_\alpha(\gamma(1)))$, where d is the metric on T_α. More generally, extend μ so that

$$\mu(\gamma_1 + \gamma_2) = \mu(\gamma_1) + \mu(\gamma_2),$$

for all piecewise transverse paths γ_1, γ_2.

3. The Combination of Actions on Trees.

Two important constructions in the category of groups are the free product with amalgamation and the HNN-extension. In this section we show in the category of group actions on trees the free product with amalgamation and the HNN-extension exist. The reader should notice the analogy with The Klein-Maskit Combination Theorems for Kleinian groups [Ma].

The following is obvious.

3.1 Lemma. *Let T_0, T_1 and T_2 be trees and $\phi_1 : T_0 \to T_1$ and $\phi_2 : T_0 \to T_2$ be monomorphisms with closed image. Then $T = (T_1 \coprod T_2)/(\phi_1 = \phi_2)$ is a tree.* □

3.2 Definition. Let $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$, be group actions on trees and $u_1 : \mathcal{T}_0 \to \mathcal{T}_1$ and $u_2 : \mathcal{T}_0 \to \mathcal{T}_2$ be monomorphisms. A *free product with amalgamation* is a commutative diagram

$$\begin{array}{ccc} & \mathcal{T}_0 & \\ u_1 \swarrow & & \searrow u_2 \\ \mathcal{T}_1 & & \mathcal{T}_2 \\ v_1 \searrow & & \swarrow v_2 \\ & \mathcal{T} & \end{array} ,$$

such that for any commutative diagram

$$\begin{array}{ccc} & \mathcal{T}_0 & \\ u_1 \swarrow & & \searrow u_2 \\ \mathcal{T}_1 & & \mathcal{T}_2 \\ v_1' \searrow & & \swarrow v_2' \\ & \mathcal{T}' & \end{array} ,$$

there is a unique $u : \mathcal{T} \to \mathcal{T}'$ with $u \circ v_i = v_i'$, $i = 1, 2$.

3.3 Theorem. *Let $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$ be group actions on trees and $u_1 : \mathcal{T}_0 \to \mathcal{T}_1$ and $u_2 : \mathcal{T}_0 \to \mathcal{T}_2$ be monomorphisms with closed image. Then there is a unique free product with amalgamation*

$$\begin{array}{ccc} & \mathcal{T}_0 & \\ u_1 \swarrow & & \searrow u_2 \\ \mathcal{T}_1 & & \mathcal{T}_2 \\ v_1 \searrow & & \swarrow v_2 \\ & \mathcal{T} & \end{array} .$$

Furthermore, v_i, $i = 1, 2$, is a monomorphism.

Proof. Suppose $\mathcal{T}_i = (G_i, T_i, \rho_i)$, $i = 0, 1, 2$. Let $\mathcal{X}_i = (X_i, (\mathcal{F}_i, \mu_i))$ be the realization of $\mathcal{T}_i$,

$i = 0, 1, 2$. By Proposition 2.6 the morphism u_i induces a morphism $f_i : X_0 \to X_i$, $i = 1, 2$. Let $\mathcal{X}_0 \times [1,2] = (X_0 \times [1,2], (\mathcal{F}_0, \mu_0) \times [1,2])$. So also $\mathcal{X}_0 \times [1,2]$ is dual to $\mathcal{T}_0$.

Define $X = (X_1 \coprod (X_0 \times [1,2]) \coprod X_2)/\sim$, where $(x,1) \sim f_1(x)$ and $(x,2) \sim f_2(x)$. And define a measured foliation $(\mathcal{F}, \mu)$ on X to extend the measured foliations on X_0, X_1, X_2 in the obvious way. Then set $\mathcal{X} = (X, (\mathcal{F}, \mu))$. We now show that $\mathcal{X}$ is dual to an action on a tree.

Clearly $\pi_1 X = G_1 *_{G_0} G_2$. By the Bass-Serre theory [Se] this splitting of $\pi_1 X$ corresponds to an action on a simplicial tree $\pi_1 X \times R \to R$. One may orient the edges of R so that for each edge e the stablizer of e is a conjugate of G_0 and the stablizer of $\partial_i e$ is a conjugate of G_i, $i = 1, 2$. There is a obvious equivariant map $q : \tilde{X} \to R$. Let $\mathcal{T}_e$ be dual to $q^{-1}(e)$, $\mathcal{T}_{\partial_i e}$ be dual to $q^{-1}(\partial_i e)$, and the monomorphism $\mathcal{T}_e \to \mathcal{T}_{\partial_i e}$ be induced by inclusion.

One can now define a direct limit in the category of group actions on trees. The directed set is the set of vertices and edges of R, with $e < \partial_1 e, \partial_2 e$. For each edge e associate the monomorphisms $\mathcal{T}_e \to \mathcal{T}_{\partial_i e}$, $i = 1, 2$. By repeated applications of Lemma 3.1 the direct limit $\mathcal{T}$ exists. Clearly there is action on a tree $\mathcal{T} = (G, T, \rho)$. By construction there is a quotient map $p : \tilde{X} \to T$ showing that $\mathcal{X}$ is dual to $\mathcal{T}$.

Now let $v_1' : \mathcal{T}_1 \to \mathcal{T}'$ and $v_2' : \mathcal{T}_2 \to \mathcal{T}'$ be morphisms, such that $v_1' \circ u_1 = v_2' \circ u_2$. Let $\mathcal{X}' = (X', (\mathcal{F}', \mu'))$ be the realization of $\mathcal{T}'$. By Proposition 2.6 there are morphisms $\mathcal{X}_i \to \mathcal{X}'$, $i = 1, 2$. These induce a morphism $\mathcal{X} \to \mathcal{X}'$. By Proposition 2.4 this induces a morphism $u : \mathcal{T} \to \mathcal{T}'$. Clearly $u \circ v_i = v_i'$, $i = 1, 2$. Now check that u is unique. □

Notice that the group of $\mathcal{T}$ is $G_1 *_{G_0} G_2$. By analogy with groups the free product with amalgamation of actions on trees of Theorem 3.3 will be denoted simply as $\mathcal{T} = \mathcal{T}_1 *_{\mathcal{T}_0} \mathcal{T}_2$ and $\mathcal{T}_i, i = 0, 1, 2$, will be identified with its image in $\mathcal{T}$.

Given an action on a tree $\mathcal{T} = (G, T, \rho)$ and $s \in G$, there is an automorphism $A_s : \mathcal{T} \to \mathcal{T}$ defined by $g \mapsto sgs^{-1}$ and $x \mapsto \rho(s)(x)$, for all $g \in G$ and $x \in T$.

3.4 Definition. Let $\mathcal{T}_0$ and $\mathcal{T}_1$, be group actions on trees and $u_1 : \mathcal{T}_0 \to \mathcal{T}_1$ and $u_2 : \mathcal{T}_0 \to \mathcal{T}_1$ monomorphisms. An *HNN-extension* is diagrams

$$\mathcal{T}_0 \xrightarrow{u_i} \mathcal{T}_1 \xrightarrow{v} \mathcal{T} = (G, T, \rho) \quad, i = 1, 2,$$

and $s \in G$ with $v \circ u_1 = A_s \circ v \circ f_2$, such that for any diagrams

$$\mathcal{T}_0 \xrightarrow{u_i} \mathcal{T}_1 \xrightarrow{v'} \mathcal{T}' = (G', T', \rho') \quad, i = 1, 2,$$

and $t \in G'$ with $v' \circ u_1 = A_t \circ v' \circ u_2$, there is a unique $u : \mathcal{T} \to \mathcal{T}'$ with $u \circ v = v'$ and $\omega(s) = t$.

The following is proved as Theorem 3.3.

3.5 Theorem. *Let T_0 and T_1 be group actions on trees and $u_1 : T_0 \to T_1$ and $u_2 : T_0 \to T_1$ be monomorphisms with closed image. Then there is a unique HNN-extension*

$$T_0 \overset{u_i}{\rightarrow} T_1 \overset{v}{\rightarrow} T = (G, T, \rho) \quad , i = 1, 2,$$

and $s \in G$ with $v \circ u_1 = A_s \circ v \circ u_2$. Furthermore, v is a monomorphism. □

Notice that the group of T is $(G_1 *_{G_0}, s)$. By analogy the HNN-extension of Theorem 3.4 will be denoted simply as $T = (T_1 *_{T_0}, s)$, where $v \circ u_1 = A_s \circ v \circ u_2$. Also T_1 will be identified with its image in T and T_0 will be identified with its image in T under $v \circ u_1$.

Notice that the length functions on G can be calculated from the length functions on each of the factor groups and amalgamating groups.

Recall that if G acts on T, then T^G is the fixed point set of G. The following two propositions will be used in the next section.

3.6 Proposition. *Let $G = G_1 *_{G_0} G_2$ and $(G, T, \rho) = T_1 *_{T_0} T_2$, where $T_i = (G_i, T_i, (\rho|G_i)|T_i)$, $i = 0, 1, 2$. If $G_1 \neq G_0$, then $T^{G_1} \subseteq T_1$.*

Proof. Suppose $G_1 \neq G_0$. Let $\mathcal{X} = (X, (\mathcal{F}, \mu))$ be as in the proof of Theorem 3.3. Recall the projections $q : \tilde{X} \to R$ and $p : \tilde{X} \to T$. By hypothesis each vertex of R has valence greater than or equal to two.

Let $y \in T^{G_1}$. There is some $x \in \tilde{X}$, such that $p(x) = y$.

If $q(x) = \partial_1 e$, then $y \in T_1$. If $q(x) \neq \partial_1 e$, then there is some $g \in G_1$, such that $q(g(x)) = g(q(x)) \neq q(x)$. Since $y \in T^{G_1}$, $g(p(x)) = p(g(x)) = p(x)$. Thus both $g(x)$ and x belong to the same leaf ℓ of $\tilde{\mathcal{F}}$. It follows that $\ell \cap q^{-1}(\partial_1 e) \neq \emptyset$ and $y \in T_1$. □

Again the following is proved as Proposition 3.6.

3.7 Proposition. *Let $G = G_1 *_{G_0}$ and $(G, T, \rho) = T_1 *_{T_0}$, where $T_i = (G_i, T_i, (\rho|G_i)|T_i)$, $i = 0, 1$. If $G_0 \neq G_1 \neq sG_0 s^{-1}$, then $T^{G_1} \subseteq T_1$.* □

4. An Inverse to the Combination of Actions on Trees.

In this section we prove inverses to the theorems of §3. The following example due to Michael Steiner shows that this may be difficult.

Let X be the quotient of two riemannian circles and a riemannian segment where each endpoint of the segment is identified with a point in a different circle. Let a, b be embedded loops in the different circles. Then $\pi_1 X = \langle a, b \rangle$ and the covering action $\langle a, b \rangle \times \tilde{X} \to \tilde{X}$ is an action on a tree. The splitting $\langle a, b \rangle = \langle a \rangle * \langle ab \rangle$ induces a spitting of the action, but the factor trees are not the corresponding minimal invariant subtrees. We leave it to the reader to find the correct splitting of the action.

4.1 Definition. Let $\mathcal{T} = (G, T, \rho)$, where $G = G_1 *_{G_0} G_2$. A free product with ammalgamation $\mathcal{T} = \mathcal{T}_1 *_{\mathcal{T}_0} \mathcal{T}_2$, where $\mathcal{T}_i = (G_i, T_i, (\rho|G_i)|T_i)$, $i = 0, 1, 2$, is *maximal* if for any other free product with amalgamation $\mathcal{T} = \mathcal{T}_1' *_{\mathcal{T}_0'} \mathcal{T}_2'$, where $\mathcal{T}_i' = (G_i, T_i', (\rho|G_i)|T_i')$, $i = 0, 1, 2$, if $T_i' \subseteq T_i$, $i = 0, 1, 2$, then $T_i' = T_i$, $i = 0, 1, 2$.

4.2 Theorem. *Let $\mathcal{T} = (G, T, \rho)$ be a minimal action on a tree and $G = G_1 *_{G_0} G_2$. If G_i is finitely generated and $G_i \neq G_0$, $i = 1, 2$, then there is a maximal free product with amalgamation $\mathcal{T} = \mathcal{T}_1 *_{\mathcal{T}_0} \mathcal{T}_2$, where $\mathcal{T}_i = (G_i, T_i, (\rho|G_i)|T_i)$, $i = 0, 1, 2$.*

Proof. Suppose $G = G_1 *_{G_0} G_2$. Let L be the set of all free product with amalgamations $\mathcal{T} = \mathcal{T}_1 *_{\mathcal{T}_0} \mathcal{T}_2$, where $\mathcal{T}_i = (G_i, T_i, (\rho|G_i)|T_i)$ and T_i is closed, $i = 0, 1, 2$. Define a partial order on L as follows. If $\mathcal{T} = \mathcal{T}_1 *_{\mathcal{T}_0} \mathcal{T}_2$, where $\mathcal{T}_i = (G_i, T_i, (\rho|G_i)|T_i)$ and $\mathcal{T} = \mathcal{T}_1' *_{\mathcal{T}_0'} \mathcal{T}_2'$, where $\mathcal{T}_i' = (G_i, T_i', (\rho|G_i)|T_i')$, then $\mathcal{T}_1 *_{\mathcal{T}_0} \mathcal{T}_2 \leq \mathcal{T} = \mathcal{T}_1' *_{\mathcal{T}_0'} \mathcal{T}_2'$ whenever $T_i \supseteq T_i'$, $i = 0, 1, 2$.

It will be shown that L has a maximal element. Firstly, L is not empty, for $\mathcal{T} = \mathcal{T}_1 *_{\mathcal{T}_0} \mathcal{T}_2$, where $\mathcal{T}_i = (G_i, T, \rho|G_i)$, $i = 0, 1, 2$.

Secondly, each chain in L has an upper bound for the following reason. Let $C = \{\mathcal{T}_1^\alpha *_{\mathcal{T}_0^\alpha} \mathcal{T}_2^\alpha\}_{\alpha \in A}$ be a chain, where $\mathcal{T}^\alpha = \mathcal{T}_1^\alpha *_{\mathcal{T}_0^\alpha} \mathcal{T}_2^\alpha$. Define $T_i = \cap T_i^\alpha$, for $i = 0, 1, 2$.

Since G_1 is finitely generated, it either has a unique minimal invariant subtree or a fixed point. In the former case each T_1^α contains the minimal invariant subtree. In the latter case Proposition 3.6 implies each $T_1^\alpha \supseteq T^{G_1}$. Thus $T_1 \neq \emptyset$. Similarly $T_2 \neq \emptyset$ and it easily follows that $T_0 \neq \emptyset$. Clearly $\mathcal{T}_1 *_{\mathcal{T}_0} \mathcal{T}_2$ is well defined and there is a morphism $\phi : \mathcal{T}_1 *_{\mathcal{T}_0} \mathcal{T}_2 \to \mathcal{T}$.

It will be shown that this is an isomorphism. To see that ϕ is a monomorphism is easy. It follows from the fact that each $\mathcal{T}_1^\alpha *_{\mathcal{T}_0^\alpha} \mathcal{T}_2^\alpha \to \mathcal{T}$ is a monomorphism. That it is an isomorphism follows from minimality.

By Zorn's Lemma L contains a maximal element.

□

4.3 Definition. Let $\mathcal{T} = (G, T, \rho)$, where $G = (G_1 *_{G_0}, s)$. An HNN-extension $\mathcal{T} = (\mathcal{T}_1 *_{\mathcal{T}_0}, s)$, where $\mathcal{T}_i = (G_i, T_i, (\rho|G_i)|T_i)$, $i = 0, 1, 2$, is *maximal* if for any other HNN-extension $\mathcal{T} = (\mathcal{T}_1' *_{\mathcal{T}_0'}, s)$, where $\mathcal{T}_i' = (G_i, T_i', (\rho|G_i)|T_i')$, $i = 0, 1$, if $T_i' \subseteq T_i, i = 0, 1$, then $T_i' = T_i, i = 0, 1$.

4.4 Theorem. *Let $\mathcal{T} = (G, T, \rho)$ be a minimal action on a tree and $G = (G_1 *_{G_0}, s)$. If G_1 is finitely generated and $G_0 \neq G_1 \neq sG_0s^{-1}$, then there is a maximal HNN-extension $\mathcal{T} = (\mathcal{T}_1 *_{\mathcal{T}_0}, s)$, where $\mathcal{T}_i = (G_i, T_i, (\rho|G_i)|T_i)$, $i = 0, 1$.*

□

5. Examples of Measured Foliations Dual to Actions on Trees.

The first example of the fundamental group of a closed surface with negative Euler characteristic acting freely on a tree is due to Morgan and Shalen [M-S4]. They prove each closed surface of Euler characteristic less than -1 has a measured, geodesic lamination, such that each leaf and each complementary component is simply connected. They then show that the lamination is dual to a free action on a tree.

Walter Parry has also constructed free actions of both orientable and non-orientable closed surfaces on trees [Pa]. He uses number theory rather than laminations to prove the action is free.

We show how to get just one such example using the methods of §3.

5.1 Example. *Let F be an orientable, closed surface of genus two. Then there is a free action on an $\mathbb{R}$-tree $(\pi_1 F, T, \rho)$ which is dual to a measured foliation $(F, (\mathcal{F}, \mu))$.*

Let each $\langle a, b\rangle$ and $\langle c, d\rangle$ be free groups of rank two.

Let X_1 and X_2 each be the wedge of two riemannian circles of circumference 1. Fix the wedge point as the base point in each $X_i, i = 1, 2$. Then $\pi_1 X_1 = \langle a, b\rangle$, where a, b are the standard generators of each of the circles. Similarly $\pi_1 X_2 = \langle c, d\rangle$, where c, d are the standard generators of each of the circles.

For $i = 1, 2$, endow X_i with a measured foliation $(\mathcal{F}_i, \mu_i)$, where each point of X_i is a leaf and the measure is given by arc length. Clearly $(X_i, (\mathcal{F}_i, \mu_i))$ is dual to the covering action $\pi_1 X_i \times \tilde{X}_i \to \tilde{X}_i$, which we call $\mathcal{T}_i$, $i = 1, 2$. Notice that each of $[a, b]$ and $[c, d]$ have translation length 4.

Let X_0 be a riemannian circle of circumference 4 and let $\pi_1 X_0 = \langle t\rangle$. Endow X_0 with a measured foliation $(\mathcal{F}_0, \mu_0)$, where again each point of X_0 is a leaf and the measure is given by arc length. Again $(X_0, (\mathcal{F}_0, \mu_0))$ is dual to the covering action $\pi_1 X_0 \times \tilde{X}_0 \to \tilde{X}_0$, which we call $\mathcal{T}_0$. Notice that t has translation length 4.

So the homomorphisms

$$t \mapsto [a, b], t \mapsto [c, d]$$

induce equivariant morphisms

$$f_1 : \tilde{X}_0 \to \tilde{X}_1, f_2 : \tilde{X}_0 \to \tilde{X}_2.$$

These in turn induce free products with amalgamation $\mathcal{T} = \mathcal{T}_1 *_{\mathcal{T}_0} \mathcal{T}_2$ which are parameterized by $\mathbb{R}$.

We will use a foliation on F to see that the action is free for all but countably many choices of f_1 and f_2.

As in the proof of Theorem 3.3 let $X = (X_1 \coprod (X_0 \times [1, 2]) \coprod X_2)/\sim$, where $\sim$ is induced by the above morphisms. And let $(\mathcal{F}, \mu)$ be a measured foliation on X which extends the measured foliations on X_0, X_1, X_2 in the obvious way. Clearly $\pi_1 X = \langle a, b\rangle *_{\langle t\rangle} \langle c, d\rangle$ and $\mathcal{X} = (X, (\mathcal{F}, \mu))$ is dual to $\mathcal{T}$. Also X is an orientable, closed surface of genus two.

Fix embeddings $X_1, X_2 \to F$, such that $F - X_1 - X_2$ is an open annulus and $[a, b]$ is isotopic to $[c, d]$. Fix an isotopy so as to make the identification $\pi_1 X = \pi_1 F$. Then there is an embedding $X \to F$ which is unique upto isotopy. This induces a measured foliation on F.

Let $\tilde{F}$ be the universal cover of F and let $(\tilde{\mathcal{F}}, \tilde{\mu})$ lift the foliation. Clearly the action on the tree is free if and only if no leaf of $\tilde{\mathcal{F}}$ is left invariant by a non-trivial element of $\pi_1 F$. Or equivalently the action is free if and only if x and $g(x)$ lie on distinct leaves of $\tilde{\mathcal{F}}$ for any x on a lift of X_1 (or X_2) and any $g \in G - \{1\}$.

To see that the action is free for all but countably many choices of f_1, f_2 is easy. Any two distinct translates of T_1 (or T_2) in T meet in at most a segment. Equivalently, any two lifts of X_1 (or X_2) in $\tilde{F}$ share the same leaves of the foliation in at most a segment. Now changing the morphisms f_1, f_2 rotates the leaves of $\tilde{\mathcal{F}}$, but the lifts of X_1 or X_2 stay fixed. Because F is orientable one may perturb f_1 and f_2 arbitrarily little to avoid fixed points. □

The above example generalizes to give for any orientable, closed surface of negative Euler characteristic a measured foliation and a dual free action. The non-orientable case is more delicate.

One should notice that Example 5.1 is not that special. By [Sk] every free action of a closed surface group on a tree is dual to a measured lamination. Given a measured lamination $(\mathcal{L}, \mu)$ on a closed surface F there is a simple, closed curve C which meets every leaf of $\mathcal{L}$ essentially and transversely. It is easy to see that the induced lamination of the surface $F - C$ is dual to a simplicial tree. Thus every free action of a closed surface group on a tree is either a free product with amalgamation or an HNN-extension of simplicial actions.

5.2 Example. *Let F be an orientable, closed surface of genus two. Then there is a non-free action on an $\mathbb{R}$-tree $(\pi_1 F, T', \rho')$ which is the image of the above free action $(\pi_1 F, T, \rho)$.*

Let X_2' be the quotient of two riemannian circles of circumference 1 and $\frac{1}{2}$, respectively and a segment of length $\frac{1}{4}$, where each endpoint of the segment is identified with a point on different circles.

Endow X_2' with a measured foliation $(\mathcal{F}_i, \mu_i)$, where each point of X_i is a leaf and the measure is given by arc length. Then there is an obvious projection $X_2 \to X_2'$ which is a morphism.

Let X' be the amalgam resulting from X_0, X_1 and X_2' with dual $(\pi_1 F, T', \rho')$. Clearly there is a morphism $(\pi_1 F, T, \rho) \to (\pi_1 F, T', \rho')$. Notice that X' is not a surface. In fact the morphism $X_0 \to X_2'$ is 4 to 1 on a set of positive measure. It follows from either [Mo] or [Sk] that $(\pi_1 F, T', \rho')$ is not free. In fact, for almost every point of T' the stablizer contains a free group of rank two. □

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