THE NIELSEN-THURSTON CLASSIFICATION AND AUTOMORPHISMS OF A FREE GROUP II

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We elaborate the techniques of [Se1] to construct a (canonical) hierarchical decomposition of a free group with respect to a given automorphism of it. Our decomposition generalizes the Nielsen-Thurston classification for automorphisms of surfaces and, from our point of view, it is also its analogue for automorphisms of free groups.

Using train tracks and invariant laminations, Thurston has developed a whole theory to understand the dynamics and geometry of diffeomorphisms of surfaces ([Th], [Ca-Bl]). By introducing a combinatorial analogue of train tracks M. Bestvina and M. Handel [Be-Ha] have managed to analyze irreducible automorphisms of a free group, and using this analysis to bound the rank of the fixed subgroup of an automorphism by the rank of the ambient group, which was known before as the Scott conjecture.

In [Se1] we introduced a dynamical-algebraic commutative diagram associated with a (limit) action of the ambient free group F_n on some real tree. This commutative diagram allows one to interpret dynamical invariants of the limit action in terms of algebraic properties of the automorphism in question and vice versa. Using this diagram we showed how to obtain the Nielsen-Thurston classification of automorphisms of surfaces on the algebraic level and some generalized versions of the Scott conjecture.

In this paper we make an extensive use of the dynamical-algebraic commutative diagram and the classification of stable actions of finitely presented groups on real trees ([Ri],[Be-Fe1]) to construct a (canonical) hierarchical decomposition of a free group associated with an automorphism of it, which is, from our point of view, the analogue of the Nielsen-Thurston classification for automorphisms of free groups. It is also our belief that our hierarchical decomposition should serve as a complementary to the Bestvina-Handel train tracks and invariant laminations in studying the dynamics and combinatorics of general (rather than irreducible) automorphisms of a free group.

The results of [Se1] are sufficient to understand all periodic conjugacy classes with respect to a given automorphism. Our whole approach to constructing the hierarchical decomposition is based on understanding the set of periodic conjugacy classes of free factors under the action of the automorphism in question. Mainly for expository reasons, since our basic notions are slightly more technical when the

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given automorphism admits periodic conjugacy classes, we construct the hierarchical decomposition for automorphisms with no periodic conjugacy classes in this paper, and in a continuation one we use the graph of groups Λ_φ, constructed in section 4 of [Se1] to prove the Scott conjecture, in order to modify our notions and generalize the construction of the hierarchical decomposition to hold for all automorphisms of a free group.

One should note, that although the hierarchical decomposition we construct is canonical, it is possible to use the results and techniques of this paper to get some closely related variations of our hierarchical decomposition. This hold for both automorphisms with no periodic conjugacy classes and for the generalization to every automorphism. It was our choice to construct it in what seemed to us as the most "compact" way, but it is possible to get slightly finer versions which may be needed in some specific situations. To our understanding, the results we obtain should enable one to get these variations without much difficulty.

We start by recalling the preliminaries needed for the techniques we use. In addition to the tools and results of the previous paper in this sequence [Se1], we quote a theorem of F. Paulin [Pa2] that allows one to replace the bi-Lipschitz equivariant map appears in our commutative diagram with an equivariant dilatation, and a theorem of D. Gaboriau and G. Levitt [Ga-Le] on the finiteness of orbits of germs of edges issuing from branching points of a real tree under a stable action of a free group. These last two results are useful when one uses our commutative diagram to interpret dynamical invariants in algebraic terms and vice versa.

We call a non-cyclic free factor P in F_n a *periodic factor* with respect to an automorphism φ if the conjugacy class of P is periodic under φ, and P contains no periodic conjugacy classes under the action of φ. We call a periodic factor *irreducible* if it contains no proper periodic factors.
In section 2 we define *Dehn* and *irreducible extensions* of periodic factors with respect to automorphisms with no periodic conjugacy classes and study their basic properties. Although these notions are well known (cf. [Be-Ha]), we prefer to define them in connection with the action of the extensions on certain simplicial trees associated with them, which is more appropriate for our techniques.

In section 3 we study the algebraic connections between "distinct" periodic factors, i.e., we study finite collections of periodic factors $P_1, \dots, P_k$ so that P_i intersects trivially every conjugate of P_j for every $j \neq i$. We do that by defining a new metric, associated with the distinct periodic factors in question, on the Cayley graph of F_n and showing that certain geodesics in the new metric are quasi-geodesics in the standard word metric. This allows us to obtain our commutative diagram from actions of the ambient group F_n on its Cayley graph equipped with the new metric, which enables us to get the main theorem of the section:

Theorem 3.11 *Let $P_1, \dots, P_k$ be periodic factors with respect to an automorphism φ of F_n with no periodic conjugacy classes. Suppose every conjugate of P_i intersects P_j trivially for $i \neq j$. Then there exist conjugates $\hat{P}_1, \dots, \hat{P}_k$ of $P_1, \dots, P_k$ so that $< \hat{P}_1, \dots, \hat{P}_k >= \hat{P}_1 * \dots * \hat{P}_k$ is a free factor in F_n.*

In the case of automorphisms with no periodic conjugacy classes, irreducible factors form the basic level of the hierarchical decomposition. Since non-conjugate irreducible factors intersect trivially, theorem 3.11 shows, in particular, that there are finitely many conjugacy classes of irreducible factors with respect to a given automorphism (lemma 5.1). Hence, having proved theorem 3.11, the basic level of the decomposition for automorphisms with no periodic conjugacy classes is set. To climb through the hierarchy of the decomposition we need to study the algebraic structure of subgroups generated by sets of "distinct" periodic factors (in the above sense) and an irreducible extension of subfactors of them which we call *irreducible bricks*.

In section 4 we apply the techniques of section 3 to study the algebraic structure of irreducible bricks (lemma 4.6), and to show that irreducible bricks are periodic factors (theorem 4.7). We further study the algebraic structure of subgroups generated by sets of "distinct" periodic factors and finite collections of irreducible extensions of their subfactors which we call *irreducible chambers* (definition 4.16). We show that irreducible chambers can be represented as iterative sequences of irreducible bricks which allows us to get their algebraic structure and to show they are periodic factors (theorem 4.17). Having studied irreducible chambers, we conclude, in particular, that a given set of distinct periodic factors can have only finitely many pairwise non-conjugate irreducible extensions of subfactors. We call the subgroup generated by a set of distinct periodic factors and all irreducible extensions of their subfactors, the *irreducible closure* of the set of periodic factors (definition 4.19).

The results and notions contained in sections 2-4 are the tools needed for the construction of the hierarchical decomposition for automorphisms with no periodic conjugacy classes in section 5. To define the decomposition we introduce some (canonical) periodic factors which we call *huts* and *blocks*. To each hut and each block there is an associated *level*, the number of levels is bounded by the rank of the ambient group F_n, and for each level there are only finitely many huts and blocks. The level 1 huts are the irreducible factors with respect to φ and the (unique) maximal level block is the ambient group F_n.

Huts and blocks are defined iteratively, i.e., the huts and blocks of level $\ell + 1$ are obtained from the set of blocks of level ℓ. The set of huts and the set of blocks of a given level are, in particular, finite sets of "distinct" periodic factors. The blocks of level ℓ are obtained from huts of the same level by an operation which we call $unifying\ periodic\ factors$ (definition 5.2). Huts of level $\ell + 1$ are defined (canonically) as subgroups generated by irreducible closures and generalized Dehn closures of subsets of blocks of level ℓ (definition 5.4).

The entire collection of huts and blocks of all levels and the way they are constructed is basically our hierarchical decomposition. The decomposition remains invariant under raising φ to a power and composing it with an inner automorphism, hence, it is really associated with (a cyclic subgroup of) outer automorphisms. We further show that every irreducible extension with respect to φ appears as one of the (canonical) irreducible extensions used to define the blocks of some level (theorem 5.6) and relate the hierarchical decomposition of the ambient group with that of a periodic factor in it (theorem 5.7). In principal, it is possible to use the hierarchical decomposition to classify all the periodic factors with respect to a given automorphism.

In a continuation paper we generalize the whole construction to hold for automorphisms with periodic conjugacy classes. We begin by erasing all edges with trivial stabilizers from the graph of groups Λ_φ, constructed in section 4 of [Se1], and call the fundamental group of each connected component a *stone* with respect to φ. We say that a free factor C in F_n is a *periodic component*, if its conjugacy class is periodic under φ and if each stone in F_n can either be conjugated into C or it intersects trivially every conjugate of C. We then show that under (minor) appropriate modifications all the results proved in sections 2-4 in this paper for periodic factors and their extensions hold for periodic components, which allow us to construct the hierarchical decomposition for general automorphisms. In general we define the huts of level 1 to be the set of irreducible factors together with the set of stones. Having modified the list of huts of level 1, the iterative construction of huts and blocks of higher levels is identical with the one introduced here for automorphisms with no periodic conjugacy classes.

We see the entire work presented in this paper as a complementary to our work on the JSJ decomposition ([Se2],[Ri-Se2]). We would like to thank Karen Vogtmann who suggested to us (in Oberwolfach 1992) that our construction of the JSJ decomposition "must" have applications to automorphisms of a free group. In a forthcoming paper of E. Rips and the author [Ri-Se3], we combine some of the techniques presented here with those used to construct the JSJ decomposition in order to study spaces of solutions to equations in a free group.

1. A Dynamical-Algebraic Commutative Diagram and its Basic Conservation Laws.

From a sequence of actions of a hyperbolic group on its Cayley graph via powers of a non-periodic automorphism, it is possible to extract a subsequence converging to an action of the group on a real tree by a theorem of F. Paulin [Pa1]. Having the action we have constructed a commutative diagram which allows one to relate algebraic assumptions with dynamical assertions and vice versa. This commutative diagram is the basis for our whole approach to the study of automorphisms of a free group. In the first paper in this sequence [Se1], we have used this commutative diagram together with Rips' classification of (stable) actions of groups on real trees to obtain the Nielsen-Thurston classification of automorphisms of surfaces, and a generalized version of the Scott conjecture for automorphisms of a free group.

In this introductory section we review the basic tools and results which already appear in [Se1]. In the following sections we elaborate and extend the list of conservation laws for our basic commutative diagram which will allow us to obtain our hierarchical decomposition. We start the section by reviewing the construction of an action of a free group on a real tree from a converging subsequence of powers of a non-periodic automorphism. We continue by stating Rips' structure theorem which gives us the essential tools to analyze the obtained action later in the sequel. We proceed by introducing the dynamical-algebraic commutative diagram of [Se1], and conclude this section by reviewing the basic conservation laws for this diagram.

Let $\Gamma \ = < G|R > = < g_1, \ldots , g_t \big| r_1, \ldots , r_s >$ be a torsion-free δ-hyperbolic group, X its Cayley graph with respect to the given presentation and φ an infinite order automorphism in Out (Γ). Since φ is not a periodic automorphism the t-tuple $(\varphi^{m_1}(g_1), \ldots , \varphi^{m_1}(g_t))$ is not conjugate to the t-tuple $(\varphi^{m_2}(g_1), \ldots , \varphi^{m_2}(g_t))$ for $m_1 \neq m_2$. For each m we pick an element $\gamma_m \in \Gamma$ which is translated minimally under the action of $\varphi^m(g_1), \ldots , \varphi^m(g_t)$ and set μ_m to be that minimal translation, i.e.:

$$\mu_m \ = \ \max_{1 \leq j \leq t} \big| \gamma_m \ \varphi^m(g_j) \, \gamma_m^{-1} \big| \ = \ \min_{\gamma \in \Gamma} \ \max_{1 \leq j \leq t} \big| \gamma \, \varphi^m \, (g_j) \gamma^{-1} \big|$$

Picking γ_m we define $\hat{\varphi}_m$ to be the automorphism given by $\hat{\varphi}_m(\gamma) = \gamma_m \varphi^m(\gamma)\gamma_m^{-1}$. Since $\hat{\varphi}_m$ is determined by the image of the generators $\{g_j\}_{j=1}^t$ and since these images are not conjugate for $m_1 \neq m_2$, necessarily $\mu_m \to \infty$. Let $\{(\mathrm{X}_m,\, id.)\}_{m=1}^{\infty}$ be the pointed metric spaces obtained from the Cayley graph X by dividing the metric d_{X} by $\mu_m \cdot (\mathrm{X}_m,\, id)$ is endowed with a left isometric action of Γ via $\hat{\varphi}_m$. At this stage we can apply the following theorem.

Theorem 1.1 ([Pa1], **2.3**) *Let $\{\mathrm{X}_m\}_{m=1}^{\infty}$ be a sequence of δ_m-hyperbolic spaces with $\delta_\infty = \underline{\lim}\, \delta_m < \infty$. Let G be a countable group isometrically acting on X_m. Suppose that for each m there exists a base point u_m in X_m such that for every finite subset P of G, the union of the geodesic segments between the images of u_m under P is compact and these unions are a sequence of totally bounded metric spaces. Then there is a subsequence converging in the Gromov topology to a $50\,\delta_\infty$-hyperbolic space X_∞ endowed with an isometric action of G.*

Our pointed metric spaces $(\mathrm{X}_m,\, id.)$ clearly satisfy the assumptions of the theorem (see [Pa1] for details) and they are $\frac{\delta}{\mu_m}$ hyperbolic, hence, there exists a subsequence $\{(\mathrm{X}_{m_k},\ id.)\}_{k=1}^{\infty}$ converging into a real tree $Y = \mathrm{X}_\infty$ which is endowed with an isometric action of Γ. By our construction Y is minimal under the action of Γ, i.e., Y contains no Γ-invariant proper subtree, and in particular the action of Γ on Y is non-trivial.

To analyze the action of Γ on the real tree Y, we need to study some of its basic properties. We start by showing the action is small and stable which will allow us to use Rips' classification of such actions in the sequel. The elementary properties we need are standard and appear in [Ri-Se2].

Proposition 1.2 ([Ri-Se2], **4.1 - 4.2**) *With the notation above:*

(i) *Stabilizers of segments of Y are either trivial or cyclic.*
(ii) *Stabilizers of tripods (convex hull of 3 points which are not on a segment) are trivial.*
(iii) *Let $[y_1,\, y_2] \subset [y_3, y_4]$ be segments of Y and assume $stab\big([y_3,\, y_4]\big) \neq 1$. Then $stab\big([y_1,\, y_2]\big) = stab\big([y_3,\, y_4]\big)$.*

Proposition 1.2 shows the action of Γ on the real tree Y is stable, so it enables analyzing the action using Rips' classification of stable actions of f.p. groups on real trees. In ([Ri] and [Be-Fe1]) the real tree Y is divided into distinct components, where on each component a subgroup of Γ acts according to one of several canonical types of actions. We bring the version of this analysis appears in the appendix of [Ri-Se2] and in ([Se3],3) which is going to be used extensively in the later sections. For the notions, the basic definitions, and the proof of the theorem below we refer the interested reader to [Ri-Se2], [Be-Fe1] and [Se3].

Theorem 1.3 (cf. ([Ri-Se2], **10.8)**,([Se3], **3.1))** *Let G be a f.g. group which admits a stable isometric action on a real tree Y. Assume the stabilizer of each tripod in Y is trivial.*

1) *There exist canonical orbits of subtrees of Y: $Y_1, \ldots Y_k$ with the following properties:*
   (i) gY_i intersects Y_j at most in one point if $i \neq j$.
   (ii) gY_i is either identical with Y_i or it intersects it at most in one point.

(iii) The action of stab (Y_i) on Y_i is either discrete or it is of axial type or IET type.

2) *The group G admits a (canonical) graph of groups with:*
   (i) Vertices corresponding to branching points with non-trivial stabilizer in Y.
   (ii) Vertices corresponding to orbits of the canonical subtrees $Y_1, \dots, Y_k$ which are of axial or IET type. The groups associated with these vertices are conjugates of the stabilizers of these components. To a stabilizer of an IET component there exists an associate 2-orbifold. All boundary components and branching points in this associated 2-orbifold stabilize points in Y. For each such stabilizer we add edges that connect the vertex stabilized by it and the vertices stabilized by its boundary components and branching points.
   (iii) A (possible) vertex stabilized by a free factor of G and connected to the other parts of the graph of groups by a unique edge with trivial stabilizer.
   (iv) Edges corresponding to orbits of edges between branching points with non-trivial stabilizer in the discrete part of Y with edge groups which are conjugates of the stabilizers of these edges.
   (vi) Edges corresponding to orbits of points of intersection between the orbits of $Y_1, \dots, Y_k$.

Having theorem 1.3 we have already shown in [Se1], that the limit real tree obtained from a sequence of powers of a non-periodic automorphism of a hyperbolic group contains no axial components isometric to a real line.

Proposition 1.4 ([Se1],1.4). *With the notation above:*

(i) *Y does not contain a minimal axial component isometric to a real line.*

(ii) *Stabilizers of non-degenerate segments which lie in the complement of the discrete parts of Y are trivial. Stabilizers of segments in the discrete components of Y are trivial or maximal cyclic.*

Having a subsequence of powers of an automorphism φ of a hyperbolic group Γ converging into a real tree Y with the above properties, we are able to introduce our commutative diagram which is the basis for our approach to the study of the dynamics of automorphisms of hyperbolic groups and in particular those of a free group.

Proposition 1.5 ([Se1],1) *Let $\{\psi_k | \psi_k = \hat{\varphi}_{m_k} = \gamma_{m_k} \varphi^{m_k} \gamma_{m_k}^{-1}\}_{k=1}^{\infty}$ be a subsequence of automorphisms obtained by theorem 1.1, namely a subsequence for which the metric spaces $\{(\mathrm{X}_{m_k}, id)\}_{k=1}^{\infty}$ equipped with a left isometric action of Γ via ψ_k converges into a real tree Y equipped with a left isometric Γ action. Let $(\mathrm{X}^1_{m_k}, id.)$ be the pointed metric space $(\mathrm{X}_{m_k}, id.)$ equipped with a left isometric action of Γ via the automorphisms $\psi^1_k = \psi_k \circ \varphi$. Then the sequence of pointed metric spaces $\{(\mathrm{X}^1_{m_k}, id.)\}_{k=1}^{\infty}$ converges in the Gromov topology on metric spaces to a pointed real tree (Y^1, y^1_0) which is isometric to the pointed real tree (Y, y_0) via an equivariant isometry $\tau : (Y^1, y^1_0) \to (Y, y_0)$ such that there exist a Γ-equivariant bi-Lipschitz map σ between Y and Y^1 and the following diagram is commutative:*

$$
(1)\qquad
\begin{array}{ccc}
\Gamma \times Y^1 & \longrightarrow & Y^1 \\
{\scriptstyle (id.,\sigma)}\Big\downarrow \;\Big\downarrow{\scriptstyle (\varphi,\tau)} & & {\scriptstyle \sigma}\Big\downarrow \;\Big\downarrow{\scriptstyle \tau} \\
\Gamma \times Y & \longrightarrow & Y
\end{array}
$$

$$\forall \gamma \in \Gamma \qquad \forall \hat{y} \in Y^1 \qquad \sigma\big(\gamma(\hat{y})\big) \;=\; \gamma\big(\sigma(\hat{y})\big);\; \tau\big(\gamma(\hat{y})\big) \;=\; \varphi(\gamma)\;\big(\tau(\hat{y})\big).$$

F. Paulin [Pa2] has shown that the bi-Lipschitz equivariant map σ is in fact an equivariant dilatation between Y and Y^1. We will use this fact in the sequel.

The commutative diagram (1) allows us to relate algebraic and dynamical properties of automorphisms of hyperbolic groups and in particular of free groups. We conclude this preliminary section by listing some of the basic conservation laws for our diagram, which helped us getting the basic dynamical-algebraic correspondences in [Se1].

Lemma 1.6 ([Se1],2.1) *With the notation of the commutative diagram (1), the automorphism φ gives a morphism between the (Rips') graph of groups associated by theorem 1.3 with the action of Γ on the real tree Y^1 and the graph of groups associated with the action of Γ on the real tree Y. In particular, the number of IET components is identical for these two graphs of groups as well as the number of orbits of points stabilized by a non-elementary subgroup of Γ, the number of orbits of edges in the discrete parts of Y and Y^1 and the number of orbits of edges stabilized by a (maximal) cyclic subgroup of Γ.*

Observing that the Rips' graph of groups associated with the action of Γ on Y is similar to the one associated with the action of Γ on Y^1, we start studying properties of the Γ-equivariant bi-Lipschitz map σ between these two Γ-real trees. Following J. Morgan [Mo] we will need the following notion:

Definition 1.7 *A subtree (or forest) T_1 of a Γ-real tree T is called mixing if for every two closed non-degenerate segments I and J in T_1, there exists a finite cover of J with closed intervals $J_1, \dots, J_n$ and elements $\gamma_1, \dots, \gamma_n \in \Gamma$ so that $\gamma_i(J_i) \subset I$ for $i = 1, \dots, n$. Note that a mixing subtree of a Γ-real tree contains, in particular, a dense orbit. In fact the orbit of every point in a mixing subtree is dense in it.*

Lemma 1.6 shows the correspondance we get from the isometry τ between the graphs of groups associated with the actions of Γ on Y and Y^1. To use these correspondance one needs to look for properties of these actions which are preserved under the action of the bi-Lipschitz equivariant map σ. The following invariants of σ are immediate from its definition.

Lemma 1.8 ([Se1],2.3) *With the notation of the commutative diagram (1):*

(i) *If T is a mixing subtree of Y^1 then $\sigma(T)$ is a mixing subtree of Y.*
(ii) *If T is a subtree of Y^1 on which Γ acts discretely, then Γ acts discretely on $\sigma(T)$.*
(iii) *If T is a subtree of Y^1 in which Γ has a dense orbit, then Γ has a dense orbit in $\sigma(T)$.*
(iv) *Let H be a subgroup of Γ that fixes a point (segment) in Y^1, then H fixes a point (segment) in Y.*

Lemma 1.8 gives us some of the elementary invariants of equivariant bi-Lipschitz maps. Using them we can start looking for parts of the graph of groups associated with the action of Γ on Y^1 which remain invariant under the action of the automorphism φ. In [Se1], in order to get the Nielsen-Thurston classification for automorphisms of surfaces and the Scott conjecture for automorphisms of a free group, we have also obtained the invariance of IET components.

Proposition 1.9 ([Se1],2.5) *Let T be an interval exchange type subtree of Y^1 and let Q be the subgroup that maps T into itself. Then $\sigma(T)$ is an interval exchange type subtree of Y and Q is its stabilizer. In particular, Q is conjugate to a (IET) vertex group in the graph of groups associated with the action of Γ on Y where boundary elements of Q are conjugate to boundary elements of this IET vertex group.*

Proposition 1.9 shows that an IET vertex group in the graph of groups associated with the action of Γ on Y^1 is a conjugate of an IET vertex group in the graph of groups associated with the action of Γ on Y. By the commuatative diagram (1) an IET vertex group in the first graph of groups is mapped by the automorphism φ to an IET vertex group in the second graph of groups. Therefore, the automorphism φ acts as a permutation on conjugacy classes of IET vertex groups of the two graphs of groups associated with the action of Γ on Y^1 and Y.

In this paper, we will be mainly interested in periodic free factors and periodic components (which will be defined in the sequel). A basic theorem which will assist us in studying the invariants of these periodic objects is the following theorem of D. Gaboriau and G. Levitt.

Theorem 1.10 ([Ga-Le],III.2) *Let F_n acts isometrically on a real tree Y. Suppose the action is minimal and satisfies the properties stated in proposition 1.2 above. Then the number of branching points and the number of orbits of germs of edges with trivial stabilizers issuing from branching points in Y are finite.*

2. Periodic Factors and their Extensions.

The commutative diagram (1) gives us a linkage between the algebraic automorphism φ and its action on the hyperbolic group Γ and a Γ-equivariant bi-Lipschitz map σ between the real trees Y and Y^1. In [Se1], in order to get the Nielsen-Thurston classification for automorphisms of surfaces, and a generalized version of the Scott conjecture for automorphisms of a free group, we mainly needed to show that stabilizers of IET components of Y^1 are mapped to stabilizers of IET components of Y by the bi-Lipschitz equivariant map σ (proposition 1.9 above). To get our hierarchical decomposition for automorphisms of a free group we will need to refine the graph of groups associated with the action of the free group on Y and Y^1 by theorem 1.3, and introduce few additional conservation laws for our diagram, which will allow us to deduce the invariance of certain parts of the refined graph of groups under our given automorphism φ.

Throughout this paper we will use the notation of the commutative diagram (1) and assume our hyperbolic group Γ is the free group $F_n = < f_1, \dots, f_n >$. We will always assume that our given automorphism φ admits no (non-trivial) periodic conjugacy classes. This case was basically not analyzed in [Se1] and the definitions and conservation laws introduced in this section are all meant to study it. In a continuation paper we slightly modify the notions and show how to combine the analysis presented in this paper with the results of [Se1] to get our hierarchical decomposition for a general automorphism.

Definition 2.1 *Let φ be an automorphism of a free group F_n. A non-cyclic free factor P of F_n is called periodic with respect to φ if P is mapped to its conjugate by some power of φ and P contains no (non-trivial) periodic conjugacy classes under the action of φ. A periodic free factor B is called irreducible if it contains no proper periodic free factors. The automorphism φ is called irreducible if the ambient group F_n is irreducible with respect to φ.*

Periodic and irreducible factors are going to serve as building blocks for our hierarchical decomposition in the case of automorphisms with no periodic conjugacy classes, and having constructed the decomposition we will be able to extract all the periodic and irreducible factors with respect to our given automorphism in this case. Note that a periodic factor has to contain an irreducible factor. Also, note that our definition of an irreducible factor, which is dynamically motivated, is slightly different from the Bestvina-Handel one [Be-Ha] since we do not allow periodic conjugacy classes in irreducible factors (the case of a pseudo-Anosov of a punctured surface is contained in the IET case, and was already handled in [Se1]).

We start this section with some very basic properties of irreducible automorphisms, all are well known and follow from Bestvina-Handel work [Be-Ha]. Like the basic analysis of pseudo-Anosovs presented in the beginning of section 3 in [Se1], our aim in presenting them is mainly showing the applicability of the commutative diagram (1) to derive algebraic information on automorphisms.

Proposition 2.2 *If φ is an irreducible automorphism then F_n acts freely on Y. Furthermore, the growth rate of elements in F_n is uniform. i.e., for any two non-trivial elements $f_1, f_2 \in F_n$ there exist constants c_1, c_2 such that for all positive m:*

$$c_1|\varphi^m(f_1)| < |\varphi^m(f_2)| < c_2|\varphi^m(f_1)|$$

and the same holds for the growth rate of their conjugacy classes.

Proof: By the construction of the decomposition Λ_φ in ([Se1],4.1), if the action of F_n on Y is not free, F_n contains either periodic free factors or periodic conjugacy classes under the action of φ, hence, if φ is irreducible the action of F_n on Y is free. The uniform growth follows by the same arguments used to prove the uniform growth in the case of pseudo-Anosov automorphisms of surfaces in part (iii) of ([Se1],3.1).

□

The intersection between two periodic free factors is either trivial or a periodic free factor, and the intersection between an irreducible factor and a periodic factor is either trivial or it is the entire irreducible factor. To analyze automorphisms we will need to study the connections between different periodic factors. To understand these connections we need the notions of *Dehn* and *irreducible extensions*.

Definition 2.3 *Let φ be an automorphism of a free group F_n, and let $P < B < F_n$ be periodic factors preserved by φ.*
Let $B = P * A$ and let Δ be a graph of groups with fundamental group B, a unique vertex group P and bouquet of circles corresponding to a set of free generators of A.

Let T be the Bass-Serre tree corresponding to Δ , let t_0 be the vertex stabilized by P in T, and let $A =< a_1, \dots , a_s >$. We say that the periodic factor B is a Dehn extension of the periodic factor P (with respect to φ) if there exists a constant λ such that for every power m and $i = 1, \dots , s$: $d_T(t_0, \varphi^m(a_i)(t_0)) \leq \lambda$. If there does not exist such constant, and every periodic factor properly contained in B can be conjugated into P, we say that B is an irreducible extension of P.

Clearly, the above definition does not depend on the specific choice of the free factor A and its generators. Before analyzing the specific structure of Dehn extensions we prove a basic property of irreducible ones.

Claim 2.4 *If B is an irreducible extension of the periodic factor P with respect to an automorphism φ, then $rank(B) - rank(P) \geq 2$.*

Proof: If $rank(B) - rank(P) = 1$ then $B = P* < b >$ for some $b \in B$. In this case every automorphism of B that preserves P must map b to $p_1 b^{\pm 1} p_2$ for some $p_1, p_2 \in P$. In particular $\varphi^m(b) = p_1(m) b^{\pm 1} p_2(m)$ and with the notation of definition 2.3 $d_T(t_0, \varphi^m(b)(t_0)) = 1$ for every m, a contradiction to the extension being irreducible.

□

The structure of Dehn extensions is well known and given by theorem 2.5 below (cf. [Be-Ha]). An alternative proof of this structure can be given using the methods of section 3 below.

Theorem 2.5 *Let B be a Dehn extension of a periodic factor P with respect to an automorphism φ. Then there exist elements $b_{1,1}, \dots , b_{1,s_1}, b_{2,1}, \dots , b_{r,s_r} \in B$ so that $B = P* < b_{1,1} > * \dots * < b_{r,s_r} >$ and for every pair i, j ($1 \leq i \leq r$; $1 \leq j \leq s_i$) there exist elements $p^1_{i,j}, p^2_{i,j} \in P$ for which $\varphi(b_{i,j}) = p^1_{i,j} b^{\pm 1}_{i,j+1} p^2_{i,j}$ where $b_{i,s_i+1} = b_{i,1}$.*

With the notation and assumptions of theorem 2.5, let B be a Dehn extension of P and let $B = P* < b_{1,1} > * \dots * < b_{r,s_r} >$ so that $\varphi(b_{i,j}) = p^1_{i,j} b^{\pm 1}_{i,j+1} p^2_{i,j}$ for some $p^1_{i,j}, p^2_{i,j} \in P$. Let Δ be a graph of groups with fundamental group B, a unique vertex group P and bouquet of circles corresponding to the elements $b_{i,j}$. Let T be the Bass-Serre tree corresponding to Δ and let t_0 be the vertex stabilized by P in T. Since $\varphi(b_{i,j}) = p^1_{i,j} b^{\pm 1}_{i,j+1} p^2_{i,j}$ and P is preserved by φ, φ acts as an (equivariant) isometry on the tree T, i.e., for any vertex $t \in T$: $d_T(t, \varphi(b)(t)) = d_T(t, b(t))$. Therefore, to the action of B on T we can associate a commutative diagram similar to the commutative diagram (1):

(2)
$$\begin{array}{ccc} B \times T & \longrightarrow & T \\ \Big\downarrow{\scriptstyle (\varphi,\nu)} & & \Big\downarrow{\scriptstyle \nu} \\ B \times T & \longrightarrow & T \end{array}$$

$\forall b \in B \qquad \forall t \in T \qquad \nu\big(b(t)\big) \ = \ \ \varphi(b)\ \big(\nu(t)\big).$

where ν is an equivariant isometry of the simplicial tree T and ν fixes t_0.

The structure of Dehn extensions given by theorem 2.5 combined with the commutative diagram (2) gives us the structure of periodic factors in Dehn extensions.

Lemma 2.6 *With the notation and assumptions of theorem 2.5 if $L < B$ is a periodic factor then L intersects non-trivially some conjugate of P.*

Proof: Let Λ_L be the Bass-Serre graph of groups L inherits from its actions on the simplicial trees T defined above. All edge stabilizers in Λ_L are trivial and vertex stabilizers are either trivial or periodic factors which are intersections of L with conjugates of P. If Λ_L contains a simple loop in which all vertex stabilizers are trivial, the commutative diagram (2) implies that the conjugacy class of the Bass-Serre generator corresponding to this simple loop is periodic. Since L does not contain any periodic conjugacy classes, there are no simple loops in Λ_L in which all vertices have trivial stabilizers. Hence, in particular, L intersects non-trivially a conjugate of P.

$\square$

Having the structure of a Dehn extension we continue by studying the subgroup generated by all periodic factors which are Dehn extensions of a given periodic factor.

Lemma 2.7 *Let φ be an automorphism of a free group F_n, let $P < B < F_n$ be periodic factors with respect to φ, and suppose φ preserves both P and B. Let $D_B(P)$ be the subgroup generated by all Dehn extensions of P which are subgroups of B. Then $D_B(P)$ is a periodic factor with respect to φ, $D_B(P)$ is preserved by φ, and $D_B(P)$ is a Dehn extension of P.*

Proof: Let $B = P * A$, let $a_1, \dots, a_s$ be a free basis for the free factor A, let Δ be a graph of groups with a unique vertex stabilized by P and a bouquet of circles corresponding to the generators of A, let T be the Bass-Serre tree corresponding to Δ, and let t_0 be the point stabilized by P in T.

If there exists a constant λ so that for every power m and every index $i = 1, \dots, s$: $d_T(t_0, \varphi^m(a_i)(t_0)) \leq \lambda$ then B is a Dehn extension of P, so suppose there is no such constant. If there is no such constant we can apply propositions 1.5, 1.6 and 1.7 of [Se1] and construct a commutative diagram (1) for the actions of the periodic factor B on limit trees V and V^1, obtained from a subsequence of the actions of B on the simplicial tree T via the automorphisms φ^m. By construction, $D_B(P)$ stabilizes points $v_0 \in V$ and $v_0^1 \in V^1$, and with the notation of the commutative diagram (1) $\sigma(v_0^1) = \tau(v_0^1) = v_0$.

Theorem 1.3 allows us to analyze the action of B on the real trees V and V^1. If V contains an IET component or a segment with non-trivial stabilizer then by theorem 4.1 of [Se1], B contains periodic conjugacy classes with respect to φ. Hence, since B is assumed a periodic free factor, and in particular it does not contain any (non-trivial) periodic conjugacy classes, V contains no IET components and the stabilizer of every non-degenerate segment in V is trivial.

Since V and V^1 contain no IET components and no segments with non-trivial stabilizers, there are finitely many orbits of points with non-trivial stabilizers in both V and V^1, and by theorem 1.3 $B = L_1 * \ldots * L_m * A$ where the L_j's are point stabilizers from distinct orbits in V, and either $m > 1$ or A is a non-trivial free factor in B.

W.l.o.g. we can assume L_1 is the stabilizer of $v_0^1 \in V^1$, so by construction it is also the stabilizer of $v_0 \in V$, $D_B(P) < L_1$ and by the commutative diagram (1) $\varphi(L_1) = L_1$. Hence, we have found a proper periodic factor L_1 in B which is preserved under φ and contains $D_B(P)$, so either L_1 is a Dehn extension of P in which case $L_1 = D_B(P)$ or we can repeat the whole process and find a proper periodic factor in L_1 having these last properties. A finite induction procedure finishes the proof of the lemma.

□

$D_B(P)$, the periodic factor generated by all Dehn extensions of the periodic factor P inside the periodic factor B is called the *Dehn closure* of P in B. If $P < D_1 < D_2 < \ldots < D_\ell < B$, D_1 is the Dehn closure of P, D_{i+1} is the Dehn closure of D_i and D_ℓ admits no Dehn extensions in B, we call D_ℓ the *generalized Dehn closure* of the periodic factor P in B and denote it $GD_B(P)$. Dehn closures are going to play an essential role in our hierarchical decomposition. The following is one of their basic properties which motivates some of our constructions in the next sections.

Lemma 2.8 *Let φ be an automorphism of a free group F_n, and let $P_1, P_2 < B$ be periodic factors with respect to φ. Assume that φ preserves B and P_1, maps P_2 to its conjugate, and P_1 intersects trivially every conjugate of P_2. Then the generalized Dehn closure $GD_B(P_1)$ intersects trivially every conjugate of $GD_B(P_2)$.*

Proof: Since $D_B(P_1)$ and $D_B(P_2)$ are periodic factors of φ by lemma 2.7, it is enough to show that $D_B(P_1)$ intersects trivially every conjugate of P_2. Suppose $D_B(P_1)$ intersects bP_2b^{-1} non-trivially. $L = D_B(P_1) \cap bP_2b^{-1}$ is a periodic factor with respect to φ, and L is a subfactor of the Dehn extension $D_B(P_1)$. Hence, lemma 2.6 implies that L intersects non-trivially a conjugate of P_1, so P_2 intersects non-trivially a conjugate of P_1, a contradiction.

□

3. Algebraic Connections Between Distinct Periodic Factors.

In the last section we introduced the basic properties of periodic factors and their Dehn and irreducible extensions. In this section we will be interested in the algebraic connections between distinct periodic factors, and in the following one we study algebraic properties of irreducible extensions of periodic subfactors of distinct periodic factors. These algebraic connections between distinct periodic factors and irreducible extensions of their periodic subfactors will allow us to understand the structure of periodic factors in section 5, and in particular the results obtained in these two sections will suffice for constructing our hierarchical decomposition for automorphisms with no periodic conjugacy classes.

Throughout this section we will assume (unless otherwise stated) that our limit real trees are obtained from sequences of actions of the ambient group F_n on its Cayley graph or on a simplicial tree with trivial edge stabilizers. We will also assume that the scaling factors used approach infinity (i.e., they do not remain bounded). If we continue to use the notation of the commutative diagram (1) (which is going to serve us intensively in this section as well), then to get the algebraic connections we are looking for, we need to look closer at the dynamics of actions of periodic factors on their minimal subtrees in Y and Y^1. We start with some very basic and useful properties of these minimal subtrees.

Proposition 3.1 *With the notation and assumptions of the commutative diagram (1) let P be a periodic factor in F_n and let Y_P a minimal subtree preserved by P. Then either Y_P is a point, or the action of P on Y_P is discrete, or the orbit of every point in Y_P is dense under the action of P.*

Proof: If Y_P is not a point and there is no dense orbit in Y_P under the action of P, theorem 1.10 on the finiteness of orbits of germs issuing from branching points implies that there must exist a segment $I \subset Y_P$ so that for every $p \in P$, which is not the identity element, $p(I)$ does not intersect I.

Since the action of P on the forest $P(I)$ is discrete, if we take out from Y_P the orbit of the segment I under the action of P we are left with finitely many orbits of connected components. If the subgroup preserving such a component is non-trivial then either it admits a dense orbit while acting on a minimal tree it preserves, or we can repeat the process. Hence, either the action of P on Y_P is discrete or Y_P contains finitely many orbits of subtrees preserved by periodic free factors, the periodic factors preserving these subtrees admit a dense orbit, and the action of P on the complement of these subtrees is discrete.

Since by theorem 1.10 there are only finitely many orbits of germs issuing from branching points in the real trees Y and Y^1, the commutative diagram (1) implies that by possibly taking a power of φ we can assume that both maps σ and τ map an orbit of germs in Y^1 into the same orbit of germs in Y.

Suppose Y (and Y^1) contain both discrete and indiscrete parts. Using [Pa2] we can assume the bi-Lipschitz P-equivariant map σ is an equivariant dilatation, and since Y contains discrete parts $\tau \circ \sigma^{-1}$ is an equivariant isometry of Y.
Let J be a germ of edges issuing from a branching point in the non-discrete part of Y. $U = \tau \circ \sigma^{-1}(J)$ is a germ of edges in the same orbit as the germ J, so $U = p(J)$ for some $p \in P$. Since J contains infinitely many branching points of germs from only finitely many orbits, there must exist an element $a \in P$ that maps a germ of edges partly overlapping with J to a germ of edges which also partly

overlapping with J. Since $\tau \circ \sigma^{-1}$ is an isometry of Y, the commutative diagram (1) implies that $\varphi(a) = pap^{-1}$, so the conjugacy class of a is periodic under the action of φ, a contradiction to P being a periodic free factor (which in particular means it contains no periodic conjugacy classes).

□

proposition 3.1 is a key point in understanding the connections between distinct periodic factors. Its first corollary is the following.

Proposition 3.2 *With the notation and assumptions of proposition 3.1:*

(i) *If P is a periodic factor which is not the entire group F_n and $f \notin P$, then Y_P meets $f(Y_P)$ at most in a single point.*

(ii) *If P_1 and P_2 are periodic factors with respect to φ, and P_1 intersects trivially every conjugate of P_2 then the minimal subtree preserved by P_1, Y_{P_1}, intersects the minimal subtree preserved by P_2, Y_{P_2}, at most in a single point.*

Proof: To prove (i) note that by taking a power of φ and composing it with an appropriate inner automorphism we get an automorphism ν that preserves P. By taking a possible power of ν, we may apply theorem 1.10 on the finiteness of orbits of germs once again, and assume that the bi-Lipschitz equivariant map σ and the equivariant dilatation τ map orbits of germs of edges issuing from branching points in Y^1_P to the same orbits in $\tau(Y^1_P) = \sigma(Y^1_P)$.

If P admits a dense orbit when acting on Y_P and P intersects $f(Y_P)$ in a non-degenerate segment, then by the finiteness of orbits of germs issuing from branching points (theorem 1.10), since τ and σ map orbits of germs of edges issuing from branching points in Y^1_P into the same orbits of germs of edges in $\tau(Y^1_P) = \sigma(Y^1_P)$, there exist some $p_1, p_2 \in P$ for which $\nu(f) = p_1 f p_2$. The same is true if Y_P is discrete since we can pick σ to be an isometry by [Pa2]. Since we assume our real trees Y and Y^1 are obtained from sequence of actions of F_n via powers of ν, either on its Cayley graph or on a simplicial tree with trivial edge stabilizers, and the rescaling factors is assumed not bounded, if $\nu(f) = p_1 f p_2$ then Y_P and $f(Y_P)$ intersect at most in one point and we get (i).

To prove (ii) note that by taking a power of φ and compose it with an appropriate inner automorphism we may assume that φ preserves P_1, and that τ and σ map orbits of germs of edges issuing from branching points in $Y^1_{P_1}$ to the same orbits of germs in $\tau(Y^1_{P_1}) = \sigma(Y^1_{P_1})$.
Suppose first that P_1 admits a dense orbit when acting on Y_{P_1}. By the finiteness of orbits of germs issuing from branching points, there exists $p_1 \in P_1$ so that $p_1(Y_{P_2})$ intersects Y_{P_2} in a non-degenerate segment. By part (i) this implies $p_1 \in P_2$, which contradicts our assumptions on P_1 intersects trivially every conjugate of P_2.
Now, suppose that both Y_{P_1} and Y_{P_2} are simplicial. Since in this case the bi-Lipschitz equivariant map is an equivariant isometry, and P_1 is preserved by φ, the commutative diagram (1) implies that there exists an element $p_1 \in P_1$ so that $\varphi(P_2) = p_1 P_2 p_1^{-1}$, which, by the construction of the real trees Y and Y^1 implies that Y_{P_1} intersects Y_{P_2} at most in a single point, and we get part (ii).

□

Propositions 3.1 and 3.2 give us the basic observations that will assist us in analyzing the connections between periodic factors algebraically. As a warmup application of these propositions we bring the following basic fact.

Lemma 3.3 *If P and Q are periodic factors with respect to an automorphism φ of F_n, every conjugate of P intersects Q trivially, and the ambient group F_n is generated by P and Q then $F_n = P * Q$.*

Proof: By possibly raising φ to a power and composing it with an inner automorphism we may assume it preserves the periodic factor P. Now, with the notation of the commutative diagram (1) let Y_P and Y_Q be the minimal subtrees preserved by P and Q in correspondence. By our assumptions and lemma 3.2, Y_P intersects Y_Q at most in a single point. If Y_P is disjoint from Y_Q or they intersect in a point that have trivial stabilizer in both Y_P and Y_Q, standard Bass-Serre theory implies that $F_n = P * Q$.
If Y_P intersects Y_Q in a point y_0 stabilized by a subgroup U then $P = (U \cap P) * P_1$ and $Q = (U \cap Q) * Q_1$ for some (possibly trivial) free factors $P_1 < P$ and $Q_1 < Q$, $U \cap P$ and $U \cap Q$ are either trivial or periodic factors in F_n, every conjugate of $U \cap Q$ intersects $U \cap P$ trivially, and $F_n = U * P_1 * Q_1$. A finite induction on the summation of the ranks of P and Q finishes the proof of the lemma.

□

The same argument generalized to finitely many periodic factors give us the following natural generalization:

Lemma 3.4 *Let $P_1, \dots, P_k$ be periodic factors with respect to an automorphism φ of F_n for which every conjugate of P_i intersects P_j trivially for $i \neq j$. Then $< P_1, \dots, P_k > = P_1 * \dots * P_k$.*

The main goal of this section is showing that a free product of some conjugates of "distinct" periodic factors form a free factor of the ambient group. A first step to achieve that goal is the following.

Theorem 3.5 *Let A and B be periodic factors with respect to an automorphism φ of F_n, suppose every conjugate of A intersects B trivially, φ preserves A and $\varphi(B) = aBa^{-1}$ for some $a \in A$. Then $H = A * B$ is a periodic factor of F_n.*

Proof: By lemma 2.8 if A intersects trivially every conjugate of B then the generalized Dehn closure of A intersects trivially every conjugate of the generalized Dehn closure of B. Since the generalized Dehn closure of a periodic factor is a periodic factor as well by lemma 2.7, to prove the theorem we may assume w.l.o.g. that both A and B admit no Dehn extensions.
If $H =< A, B >$ is the entire ambient group F_n the theorem follows from lemma 3.3, so suppose F_n is generated by A, B and the elements $f_1, \dots, f_s \in F_n$ where s is the minimal number of elements needed to be supplemented to A and B in order to generate F_n. Let G be a free group given by the free product $G = A_1 * B_1 * < u_1 > * \dots * < u_s >$ where $A_1 \simeq A$ and $B_1 \simeq B$, and let $\rho : G \to F_n$ be the natural epimorphism sending A_1 to A, B_1 to B and u_i to f_i for $i = 1, \dots, s$.

We denote by $\ell_{A,B}(g)$ the length of a normal form of an element $g \in G$. This length function on G defines a natural metric on F_n through the epimorphism ρ:

$$d_{A,B}(f, id.) \ = \ \min_{g \in \rho^{-1}(f)} \ell_{A,B}(g)$$

Clearly, $d_{A,B}(\varphi^m(a), id.) = 1$ for $a \in A$ and $d_{A,B}(\varphi^m(b), id.)$ is either 1 or 3 for $b \in B$. For the rest of the proof of the theorem we fix a basis $a_1, \dots, a_p$ for A and

$b_1, \dots, b_q$ for B. For any word w in the a's b's and f's we can associate a length $\ell_{A,B}(w)$ by naturally identifying it with a word in the group G. As the lemma below shows, one of the basic properties of our metric $d_{A,B}$ is that words achieving minimal distance are quasi-geodesics in the standard (word) metric.

Lemma 3.6 *With the notation and assumptions above, there exists a constant λ (depending only on A, B their bases and the elements f_i) so that if w is a word in the a's b's and the f_i's for which all its subwords which are exclusively a product of a's or b's are geodesics in A and B in correspondence, and w is length minimizing, i.e., $\ell_{A,B}(w) = d_{A,B}(w, id.)$, then w is a λ-quasi-geodesic in the Cayley graph of F_n equipped with the word metric.*

Proof: Since every f.g. subgroup of a free group is quasi-convex, the subgroups A, B, $H = A * B$ and $D =< f_1, \dots, f_s >$ are quasi-convex. Hence, there exists a constant λ_0 so that every subword of w that lies entirely in H or entirely in D is a λ_0-quasi-geodesic in the Cayley graph X of F_n equipped with the word metric. To prove the lemma we first need to generalize this fact to words of bounded length. Let $|w|$ denote the distance from the identity to w in the word metric.

Claim 3.7 *With the notation and assumptions of lemma 3.6 if w is a word in the a's b's and f_i's, every subword of w which is a product of exclusively a's or b's is a geodesic in A and B in correspondence, $\ell_{A,B}(w) = d_{A,B}(w, id.)$, and v is a subword of w so that $|v| \leq k$ (in the word metric), then v is a λ_k-quasi-geodesic in the word metric on X.*

Proof: Under the assumptions of the claim suppose $v = h_1 d_1 \dots h_m d_m$ where $h_i \in H$ and $d_i \in D$. Since every subword of v which lies in H is a quasi-geodesic:

$$|v| \geq \sum |h_i| - \sum |d_i| - c_k$$

Since $s < k$ and $\sum |d_i| \leq k$ the total length of the subwords h_i is bounded by: $\sum |h_i| \leq k + c_k$. Since every subword h_i is a λ_0-quasi-geodesic, if $\ell(h_i)$ denotes the length of h_i as a word in the a's and b's then $\sum \ell(h_i) \leq \lambda_0(k + c_k)$. Hence, if $\ell(v)$ denotes the length of v as a word in the a's, b's and f_i's $\ell(v) \leq \lambda_0(k + c_k) + k$ and the claim follows.

□

Now, let the maximum length of the a's b's and the f_i's in the word metric on F_n be k_1. The Cayley graph X of F_n is a tree, so let the points $r_0 = id., \dots, r_{c_w} = w$ lie on a geodesic (in the word metric) from the identity to w where $d(r_i, r_{i-1}) = 10k_1$. Since the word w represents a path from the identity to w in the tree X, there exist points $\hat{r}_0 = id., \dots, \hat{r}_{c_w} = w$ so that each $\hat{r}_i$ is a subword of w and $d(r_i, \hat{r}_i) \leq k_1$. Clearly $d(\hat{r}_i, \hat{r}_{i-1}) \leq 12k_1$.

By claim 3.7 $\ell(w) \leq 12k_1 c_w \lambda_{12k_1}$. Hence, $\ell(w) \leq \frac{12}{10}\lambda_{12k_1}|w|$ and $|w|$ is a λ-quasi-geodesic for $\lambda = \frac{12}{10}\lambda_{12k_1}$, so the proof of lemma 3.6 is completed.

□

We call a word w that satisfies the assumptions of lemma 3.6 a *restricted geodesic with respect to the metric $d_{A,B}$*. To prove theorem 3.5 we need to separate our argument. We first study the case in which the above distance function is bounded on the set $\{\varphi^m(f)\}$ for every (fixed) $f \in F_n$, and then treat the case in which it is not bounded.

Lemma 3.8 *With the notation and assumptions of theorem 3.5 suppose that there exists a constant c so that $d_{A,B}(\varphi^m(f_i), id.) < c$ for every m and every $i = 1, \dots, s$. Then there exists a power k_0 and elements $v_1, \dots, v_r \in F_n$ so that $v_j \notin H = A * B$, B and $v_1, \dots, v_r$ generate F_n, $\varphi^{k_0}(B) = a_{k_0} B a_{k_0}^{-1}$ where $a_{k_0} \in A$, and there exist non-trivial elements $b_j^1 \in B$ and $a_j^2 \in A$ for $j = 1, \dots, r$ so that:*

$$\varphi^{k_0}(v_j) = a_{k_0} b_j^1 v_j a_j^2$$

Proof: Since $d_{A,B}(\varphi^m(f_i), id.)$ remains bounded for all powers m and $i = 1, \dots, s$, a simple pigeon-hole argument shows that there exists a fixed index n_0 so that we can restrict to a subsequence of powers of φ for which the following words $w_m = \varphi^{n_m}(f_1)$ and $\hat{w}_m = \varphi^{n_m+n_0}(f_1)$, which are restricted geodesics with respect to the metric $d_{A,B}$, have the form:

$$\varphi^{n_m}(f_1) = w_m = a_{1,1}^m b_{1,1}^m \dots a_{1,i_1}^m b_{1,i_1}^m t_1 a_{2,1}^m b_{2,2}^m \dots t_2 \dots a_{k,i_k}^m b_{k,i_k}^m t_k$$

$$\varphi^{n_m+n_0}(f_1) = \hat{w}_m = \hat{a}_{1,1}^m \hat{b}_{1,1}^m \dots \hat{a}_{1,i_1}^m \hat{b}_{1,i_1}^m t_1 \hat{a}_{2,1}^m \hat{b}_{2,2}^m \dots t_2 \dots \hat{a}_{k,i_k}^m \hat{b}_{k,i_k}^m t_k$$

where the subwords t_j are words in the $f_1, \dots, f_s$, $t_j \notin A$, $t_j \notin B$ and t_j is independent of the index m, $a_{j,p}^m, \hat{a}_{j,p}^m \in A$ and $b_{j,p}^m, \hat{b}_{j,p}^m \in B$.
Since F_n is assumed to have no periodic conjugacy classes, the length of at least some of the words $a_{j,p}^m$ or $b_{j,p}^m$ must be unbounded in the word metric on F_n for fixed index $\{j,p\}$. Hence, we may pass to a further subsequence of powers of φ (still denoted n_m) so that the following subwords $w_m = \varphi^{n_m}(f_1)$ and $\hat{w}_m = \varphi^{n_m+n_0}(f_1)$, which are restricted geodesics with respect to the metric $d_{A,B}$, have the form:

$$\varphi^{n_m}(f_1) = w_m = a_{1,1}^m b_{1,1}^m \dots a_{1,i_1}^m b_{1,i_1}^m x_1 a_{2,1}^m b_{2,1}^m \dots x_2 \dots a_{k,i_k}^m b_{k,i_k}^m x_k$$

$$\varphi^{n_m+n_0}(f_1) = \hat{w}_m = \hat{a}_{1,1}^m \hat{b}_{1,1}^m \dots \hat{a}_{1,i_1}^m \hat{b}_{1,i_1}^m x_1 \hat{a}_{2,1}^m \hat{b}_{2,2}^m \dots x_2 \dots \hat{a}_{k,i_k}^m \hat{b}_{k,i_k}^m x_k$$

where the subwords x_j are independent of m, $a_{j,p}^m, \hat{a}_{j,p}^m \in A$ and $b_{j,p}^m, \hat{b}_{j,p}^m \in B$ and the length of $a_{j,p}^m$, $\hat{a}_{j,p}^m$, $b_{j,p}^m$ and $\hat{b}_{j,p}^m$ converge to ∞ in the word metric on F_n.

By lemma 3.6 the words w_m and $\hat{w}_m$ are λ-quasi-geodesics in the standard word metric on the Cayley graph X for some fixed constant λ. Since w_m is a λ-quasi-geodesic and the automorphism φ^{n_0} acts as a bi-Lipschitz equivariant map on the Cayley graph X equipped with the word metric, the words $w'_m = \varphi^{n_m+n_0}(f_1)$ given by:

$$\begin{aligned}\varphi^{n_m+n_0}(f_1) = w'_m = &\varphi^{n_0}(a_{1,1}^m)\varphi^{n_0}(b_{1,1}^m) \dots \varphi^{n_0}(x_1)\varphi^{n_0}(a_{2,1}^m)\varphi^{n_0}(b_{2,1}^m) \\ &\dots \varphi^{n_0}(x_2) \dots \varphi^{n_0}(a_{k,i_k}^m)\varphi^{n_0}(b_{k,i_k}^m)\varphi^{n_0}(x_k)\end{aligned}$$

are λ'-quasi-geodesics for some fixed constant λ'.

Since $\varphi^{n_0}(B) = a_{n_0} B a_{n_0}^{-1}$, $\hat{w}_m = w'_m = \varphi^{n_m+n_0}(f_1)$, $\hat{w}_m$ is a λ-quasi-geodesic and w'_m is a λ'-quasi-geodesic, the lengths of the subwords $a_{j,p}^m$, $b_{j,p}^m$, $\hat{a}_{j,p}^m$ and $\hat{b}_{j,p}^m$ grows to ∞, the periodic factors A and B are malnormal in F_n, every conjugate of A intersects B trivially, for every x_j for which x_j is not an element of A nor B $\varphi^{n_0}(x_j)$ must have one of the following forms depending on whether the parts of the subwords of w_m before and after x_j belong to A or B in correspondence:

(i) if $a_{j,i_j}^m x_j a_{j+1,1}^m$ is a subword of w_m then $\varphi^{n_0}(x_j) = a_j^1 x_j a_j^2$ for some $a_j^1, a_j^2 \in A$.

(ii) if $b^m_{j,i_j} x_j b^m_{j+1,1}$ is a subword of w_m then $\varphi^{n_0}(x_j) = a_{n_0} b^1_j x_j b^2_j a^{-1}_{n_0}$ for some $b^1_j, b^2_j \in B$.

(iii) if $a^m_{j,i_j} x_j b^m_{j+1,1}$ is a subword of w_m then $\varphi^{n_0}(x_j) = a^1_j x_j b^2_j a^{-1}_{n_0}$ for some $a^1_j \in A$ and $b^2_j \in B$.

(iv) if $b^m_{j,i_j} x_j a^m_{j+1,1}$ is a subword of w_m then $\varphi^{n_0}(x_j) = a_{n_0} b^1_j x_j a^2_j$ for some $b^1_j \in B$ and $a^2_j \in A$.

In case (i) the periodic factor A admits a Dehn extension, and in case (ii) the periodic factor B admits a Dehn extension, but we assumed that both A and B admit no Dehn extensions, so they can't exist. Replacing x_j with x_j^{-1} when necessary we get that for all $x_1, \ldots, x_k$ that are not elements of the periodic factors A nor B: $\varphi^{n_0}(x_j) = a_{n_0} b^1_j x_j a^2_j$. If either b^1_j or a^2_j are trivial, either A or B admit a Dehn extension which contradicts our assumptions, so both a^2_j and b^1_j are non-trivial for all j. Repeating our argument to $f_2, \ldots, f_s$ and replacing the x_j's obtained for all the f_i's with $v_1, \ldots, v_r$ we have $F_n =< A, B, v_1, \ldots, v_r >$ and the lemma follows. $\square$

Lemma 3.8 allows us to complete the proof of theorem 3.5 in case the distance function $d_{A,B}$ remains bounded. Proposition 3.9 below concludes that in this bounded case the ambient group F_n is a Dehn extension of the periodic factor $H = A * B$.

Proposition 3.9 *With the notation and assumptions above suppose there exists a constant c so that $d_{A,B}(\varphi^m(f_i), id.) < c$ for every m and every $i = 1, \ldots, s$. Then there exists a power k_0 and elements $e_1, \ldots, e_s \in F_n$ so that $F_n = A * B * < e_1 > * \ldots * < e_s >$, $\varphi^{k_0}(B) = a_{k_0} B a^{-1}_{k_0}$ where $a_{k_0} \in A$, and there exist non-trivial elements $b^1_j \in B$ and $a^2_j \in A$ for $j = 1, \ldots, s$ so that:*

$$\varphi^{k_0}(e_j) = a_{k_0} b^1_j e_j a^2_j$$

Furthermore, under the assumptions given above F_n is a Dehn extension of the periodic factor $H = A * B$ with respect to the given automorphism φ.

Proof: To prove the proposition one can either use the dynamics of the action of F_n on a limit real tree obtained from a subsequence of powers of the automorphism φ (in particular propositions 3.1 and 3.2 above) or give a combinatorial argument. Since most of this paper uses the real tree point of view we present the combinatorial approach.

With the notation of lemma 3.8 let $e_1, \ldots, e_t$ be a subset of the elements $v_1, \ldots, v_r$ which are not elements of H and which represent all double cosets Bv_jA so that e_i and e_j belong to distinct double cosets if $i \neq j$. Clearly, $\varphi^{k_0}(e_j) = a_{k_0} b^1_j e_j a^2_j$ and $\varphi^{mk_0}(e_j) = a_{mk_0} b^1_j(m) e_j a^2_j(m)$ where $\varphi^{mk_0}(B) = a_{mk_0} B a^{-1}_{mk_0}$ for every positive m. If the sequence of elements $\{b^1_j(m)\}$ (for fixed j) admits a bounded subsequence in the word metric on F_n, either A admits a non-trivial Dehn extension or $abe_j \in A$ for some $a \in A$ and $b \in B$. The first possibility contradicts our assumptions, and the second implies that $e_j \in A * B$ which contradicts our assumptions as well. Hence, the sequences $\{b^1_j(m)\}$ and $\{a^2_j(m)\}$ have no bounded subsequences with respect to the word metric. By the same argument the sequences $\{a^2_j(m) \varphi^{mk_0}(a)\}$ for a fixed $a \in A$ and $a^{-1}_{mk_0} \varphi^{mk_0}(b) a_{mk_0} b^1_j(m)$ for a fixed $b \in B$ have no bounded subsequences with respect to the word metric.

If $a^2_j(m)\varphi^{mk_0}(a)(a^2_{j'}(m))^{-1}$ for some $1 \leq j, j' \leq t$ and (fixed) $a \in A$ contains a bounded subsequence then by the same argument used above either B admits a Dehn extension or for some $\hat{a} \in A$ $e_j \hat{a} e_{j'}^{-1} \in B$. Hence, $e_j \in B e_{j'} A$ so $j = j'$ since the e_j's belong to different double cosets. Since every conjugate of A intersects B trivially, necessarily $a = 1$. So if $a \neq 1$ or $j \neq j'$ the sequence $a^2_j(m)\varphi^{mk_0}(a)(a^2_{j'}(m))^{-1}$ contains no bounded subsequence with respect to the word metric. Similarly if $b \neq 1$ or $j \neq j'$ the sequence $(b^1_j(m))^{-1} a^{-1}_{mk_0} \varphi^{mk_0}(b) a_{mk_0} b^2_{j'}(m)$ contains no bounded subsequence with respect to the word metric.

Let w be a non-trivial word in the alphabet $a_1, \ldots, a_p, b_1, \ldots, b_q, e_1, \ldots, e_t$. Combining all the observations above the length of $\varphi^{mk_0}(w)$ with respect to the word metric on F_n grows to ∞, so in particular w represents a non-trivial element in F_n and $F_n = A * B * < e_1 > * \ldots * < e_t >$, and $H = A * B$ is a free factor of F_n. Since s was the minimal number of elements needed to add to A and B in order to generate F_n necessarily $t = s$.

Let T be the Bass-Serre tree corresponding to a graph of groups with fundamental group F_n, one vertex stabilized by $H = A * B$ and bouquet of circles corresponding to $e_1, \ldots, e_s$. Let t_0 be the vertex stabilized by H in T. Since φ preserves H and the length of a sequence $d_T(\varphi^m(f), t_0)$ remains bounded for fixed $f \in F_n$, F_n is a Dehn extension of the periodic factor H according to definition 2.3.

□

To complete the proof of theorem 3.5 we need to study the case in which the distance function $d_{A,B}$ is unbounded. To analyze this case we construct a commutative diagram from the actions of F_n on its Cayley graph X equipped with the metric $d_{A,B}$.

Proposition 3.10 *With the notation and assumptions above suppose there exists an element $f \in F_n$ for which $d_{A,B}(\varphi^m(f), id.)$ is not bounded. Then there exists a periodic factor P properly contained in F_n, P is preserved by φ and $A * B = H < P$.*

Proof: Let $(X, d_{A,B})$ denote the metric space which is the Cayley graph X of the ambient group F_n equipped with the metric $d_{A,B}$ defined above. F_n acts isometrically on $(X, d_{A,B})$ by left translations.
Let $f_1, \ldots, f_n$ be a generating set for the free group F_n. If we set $\mu_m = \max_{1 \leq j \leq n} d_{A,B}(\varphi^m(f_j), id.)$ then by our assumptions, after possibly passing to a subsequence, $\mu_m \to \infty$.

Since restricted geodesics with respect to the metric $d_{A,B}$ are quasi-geodesics in the word metric by lemma 3.6, and since the rescaling factors $\mu_m \to \infty$, we can extract from the sequence of actions of F_n on the metric spaces $(X, d_{A,B})$ a subsequence converging to a real tree Y by theorem 1.1 ([Pa1],2.3). The action of F_n on Y is stable and stabilizers of segments are either trivial or maximal cyclic by the proof of proposition 1.2 ([Ri-Se2],4.1-4.2). Furthermore, by construction the subgroup $H = A * B$ is a subgroup of a point stabilizer P in Y.

Since φ acts on the metric space $(X, d_{A,B})$ as a bi-Lipschitz equivariant map, and since $\mu_m \to \infty$, we can also construct a commutative diagram (1) from a converging subsequence of actions of F_n on the metric space $(X, d_{A,B})$ via powers of the automorphisms φ. Since φ was assumed to have no periodic conjugacy classes, the commutative diagram (1) implies that Y contains no IET components and that the stabilizer of every segment in Y is trivial. Hence, the stabilizer of every point in Y is a periodic factor with respect to φ. Since H is invariant under

φ so is the point stabilizer containing H, so $H < P$, P is a periodic factor properly contained in F_n, and P is invariant under the action of φ.

$\square$

Lemma 3.3 and propositions 3.9 and 3.10 conclude the proof of theorem 3.5 by a finite induction on the rank of the ambient group F_n.

$\square$

Theorem 3.5 shows that if A and B are periodic factors so that A intersects trivially every conjugate of B and φ preserves $A*B$, then $A*B$ is a periodic factor. Its natural generalization which strongly motivates the whole construction of our hierarchical decomposition and is already sufficient to construct its basic level is the following theorem.

Theorem 3.11 *Let $A_1, \dots, A_k$ be periodic factors with respect to an automorphism φ of F_n. Suppose every conjugate of A_i intersects A_j trivially for $i \neq j$. Then there exist conjugates $\hat{A}_1, \dots, \hat{A}_k$ of $A_1, \dots, A_k$ so that $< \hat{A}_1, \dots, \hat{A}_k >= \hat{A}_1 * \dots * \hat{A}_k$ is a free factor in F_n.*

Proof: The proof of theorem 3.11 is a generalization of the proof of theorem 3.5. By lemma 2.8 we may assume w.l.o.g. that all the periodic factors $A_1, \dots, A_k$ admit no Dehn extensions. By possibly taking a power of φ we can assume that φ preserve the conjugacy classes of $A_1, \dots, A_k$ and by composing φ with an appropriate inner automorphism we may assume that φ preserves A_1.

If F_n is generated by some conjugates $\hat{A}_1, \dots, \hat{A}_k$ of $A_1, \dots, A_k$, the theorem follows from lemma 3.4, so suppose there are no k-tuple of conjugates of $A_1, \dots, A_k$ that generate the ambient group F_n, and F_n is generated by $A_1, \dots, A_k$ and the elements $f_1, \dots, f_s \in F_n$ where s is the minimal number of elements needed to be supplemented to $A_1, \dots, A_k$ in order to generate F_n. Let G be a free group given by the free product $G = B_1 * \dots * B_k * < u_1 > * \dots * < u_s >$ where $B_i \simeq A_i$, and let $\rho : G \to F_n$ be the natural epimorphism sending B_i to A_i, and u_i to f_i for $i = 1, \dots, s$.

We denote by $\ell_{A_1,\dots,A_k}(g)$ the length of a normal form of an element $g \in G$. This length function on G defines a natural metric on F_n through the epimorphism ρ:

$$d_{A_1,\dots,A_k}(f, id.) \ = \ \min_{g \in \rho^{-1}(f)} \ell_{A_1,\dots,A_k}(g)$$

Clearly, $d_{A_1,\dots,A_k}(\varphi^m(a_1), id.) = 1$ for $a_1 \in A_1$. For the rest of the proof of the theorem we fix a basis $a_{i,1}, \dots, a_{i,p_i}$ for the periodic factor A_i ($i = 1, \dots, k$). For every word w in the a's and f's we can associate a length $\ell_{A_1,\dots,A_k}(w)$ by naturally identifying it with a word in the group G. The argument used to prove lemma 3.6 naturally generalizes to restricted geodesics with respect to the metric $d_{A_1,\dots,A_k}$.

Lemma 3.12 *With the notation and assumptions above, there exists a constant λ (depending only on $A_1, \dots, A_k$ their chosen bases and the elements f_i) so that if w is a word in the $a_{i,j}$'s and the f_i's for which all its subwords which are exclusively a product of $a_{i,j}$'s from the same periodic factor A_i are geodesics in A_i, and w is length minimizing, i.e., $\ell_{A_1,\dots,A_k}(w) = d_{A_1,\dots,A_k}(w, id.)$, then w is a λ-quasi-geodesic in the Cayley graph of F_n equipped with the word metric.*

We call words w in the $a_{i,j}$'s and f_i's that satisfy the assumptions of lemma 3.12 *restricted geodesics with respect to the metric $d_{A_1,\dots,A_k}$*. Like the proof of theorem 3.5, to prove theorem 3.11 we need to separate our argument. We first study the case in which the sequence $\{d_{A_1,\dots,A_k}(\varphi^m(f),id.)\}$ is bounded for every (fixed) $f \in F_n$, and then treat the case in which it is not bounded.

Proposition 3.13 *With the notation and assumptions above suppose there exists a constant c so that for every m $d_{A_1,\dots,A_k}(\varphi^m(f_i),id.) < c$ for $i = 1,\dots,s$, and $d_{A_1,\dots,A_k}(\varphi^m(a_{i,j}),id.) < c$ for $i = 1,\dots,k$ and $j = 1,\dots,p_i$. Then there exists a power n_0 and a conjugate $\hat{A}_i$ of one of the periodic factors A_i for $i > 1$, so that $H = A_1 * \hat{A}_i$ is invariant under φ^{n_0}.*

Proof: By the same argument used in the proof of lemma 3.8, since $d_{A_1,\dots,A_k}(\varphi^m(f_i),id.)$ remains bounded for all powers m and fixed f_i for some $i = 1,\dots,s$, there exists a fixed power n_0 so that we can restrict to a subsequence of powers of φ for which the following words $w_m = \varphi^{n_m}(f_1)$ and $\hat{w}_m = \varphi^{n_m+n_0}(f_1)$, which are restricted geodesics with respect to the metric $d_{A_1,\dots,A_k}$, have the form:

$$\varphi^{n_m}(f_1) = w_m = t_1 v_1(m) t_2 v_2(m) \dots t_r v_r(m) t_{r+1}$$

$$\varphi^{n_m+n_0}(f_1) = \hat{w}_m = t_1 \hat{v}_1(m) t_2 \hat{v}_2(m) \dots t_r \hat{v}_r(m) t_{r+1}$$

where the subwords t_j may be trivial and t_j is independent of the index m, the subwords $v_j(m)$ and $\hat{v}_j(m)$ belong to the same periodic factor $A_{i(j)}$ (where the index $i(j)$ is independent of m) and the lengths of both $v_j(m)$ and $\hat{v}_j(m)$ grows to ∞ (see the proof of lemma 3.8 for a more detailed explanation of the forms of w_m and $\hat{w}_m$).

By lemma 3.12 the words w_m and $\hat{w}_m$ are λ-quasi-geodesics in the standard word metric on the Cayley graph X for some fixed constant λ. Since w_m is a λ-quasi-geodesic and the automorphism φ acts as a bi-Lipschitz equivariant map on the Cayley graph X equipped with the word metric, the words $w'_m = \varphi^{n_m+n_0}(f_1)$ given by:

$$\varphi^{n_m+n_0}(f_1) = w'_m = \varphi^{n_0}(t_1)\varphi^{n_0}(v_1(m))\varphi^{n_0}(t_2)\varphi^{n_0}(v_2(m))\dots\varphi^{n_0}(t_r(m))\varphi^{n_0}(v_r(m))\varphi^{n_0}(t_{r+1})$$

are λ'-quasi-geodesics for some fixed constant λ'.

Since φ^{n_0} preserves the conjugacy classes of the periodic factors $A_2,\dots,A_k$ $\varphi^{n_0}(v_j(m)) = u_{i(j)} v'_j(m) u_{i(j)}^{-1}$ for some $v'_j(m) \in A_{i(j)}$. Since φ preserves the periodic factor A_1, if $v_j(m) \in A_1$ then $\varphi^{n_0}(v_j(m)) \in A_1$.
Since $\hat{w}_m = w'_m = \varphi^{n_m+n_0}(f_1)$, $\hat{w}_m$ is a λ-quasi-geodesic and w'_m is a λ'-quasi-geodesic, the lengths of the subwords $v_j(m)$ grows to ∞, the periodic factors $A_1,\dots,A_k$ are malnormal in F_n, and every conjugate of A_i intersects A_j trivially for $i \neq j$, if $v_1(m) \in A_1$ and $t_1 \neq 1$ then $\varphi^{n_0}(t_1) = t_1 a_1$ for some $a_1 \in A_1$. Since $t_1 \notin A_1$ in this case, this implies that A_1 admits a Dehn extension which contradicts our assumptions, hence, if $t_1 \neq 1$ then $v_1(m) \notin A_1$.

If $v_1(m) \in A_i$ where $i \neq 1$, then $t_1 \hat{a}_i = \varphi^{n_0}(t_1) u_i a'_i$ for some $\hat{a}_i, a'_i \in A_i$, so $u_i = \varphi^{n_0}(t_1^{-1}) t_1 a_i$ for some $a_i \in A_i$. Hence, $\varphi^{n_0}(t_1 A_i t_1^{-1}) = t_1 A_i t_1^{-1}$. Since φ^{n_0} preserves both A_1 and $t_1 A_i t_1^{-1}$ for $i \neq 1$, the subgroup $H = A_1 * t_1 A_i t_1^{-1}$ is preserved by φ^{n_0} and the proposition follows in this case.

We are left with the case $t_1 = 1$ and $v_1(m) \in A_1$. If $t_2 = 1$ then $v_2(m) \in A_i$ for some $i \neq 1$ and $\varphi^{n_0}(A_i) = a_1 A_i a_1^{-1}$ for some $a_1 \in A_1$, so the subgroup $H = A_1 * A_i$ is invariant under φ^{n_0}. Hence, we may assume $t_2 \neq 1$ and clearly $t_2 \notin A_1$. If

$v_2(m) \in A_1$ then $\varphi^{n_0}(t_2) = \hat{a}_1 t_2 a'_1$ for some $\hat{a}_1, a'_1 \in A_1$, hence, A_1 admits a Dehn extension which contradicts our assumptions.
If $v_2(m) \in A_i$ for $i > 1$ then $\hat{a}_1 t_2 \hat{a}_i = a'_1 \varphi^{n_0}(t_2) u_i a'_i$ for some $\hat{a}_1, a'_1 \in A_1$ and $\hat{a}_i, a'_i \in A_i$. Hence, $u_i = \varphi^{n_0}(t_2^{-1}) a_1 t_2 a_i$ for some $a_1 \in A_1$ and $a_i \in A_i$, and the subgroup $H = A_1 * t_2 A_i t_2^{-1}$ is invariant under the action of φ^{n_0}, so the proposition follows.

□

To complete the proof of theorem 3.11 we need to study the case in which the distance function $d_{A_1,\dots,A_k}$ is unbounded. Like in the proof of theorem 3.5 (proposition 3.10) to analyze this case we construct a commutative diagram from the actions of F_n on its Cayley graph X equipped with the metric $d_{A_1,\dots,A_k}$.

Proposition 3.14 *With the notation and assumptions above suppose there exists an element $f \in F_n$ for which $d_{A_1,\dots,A_k}(\varphi^m(f), id.)$ is not bounded. Then there exist periodic factors $P_1, \dots, P_r$ ($r \geq 1$) properly contained in F_n, so that $P_1 * \dots * P_r$ is a free factor in F_n, and for each $i = 1, \dots, k$ the periodic factor A_i can be conjugated into one of the periodic factors P_j ($j = 1, \dots, r$).*

Proof: By our assumptions the conjugacy class of the periodic factor A_i is preserved by φ for $i = 1, \dots, k$. Let $u_i(m)$ be the shortest element (with respect to the word metric) in F_n for which $\varphi^m(A_i) = u_i(m) A_i u_i(m)^{-1}$. Since A_i is in particular quasi-convex and malnormal, there exist a constant λ' so that $u_i(m) a_i u_i(m)^{-1}$ is λ'-quasi-geodesic for every $m > 1$ and every non-trivial element $a_i \in A_i$ where $i = 1, \dots, k$.
Since restricted geodesics with respect to the metric $d_{A_1,\dots,A_k}$ are λ-quasi-geodesics with respect to the word metric on F_n by lemma 3.12, every restricted geodesic from the identity to $\varphi^m(a_i)$ for some non-trivial $a_i \in A_i$ has to remain in a bounded distance (in the word metric) from the λ'-quasi-geodesic $u_i(m) a_i u_i(m)^{-1}$. Hence, there exists a constant c so that for every power $m > 0$, every $i = 1, \dots, k$ and every non-trivial elements $a_i, \hat{a}_i \in A_i$:

$$\left| d_{A_1,\dots,A_k}(\varphi^m(a_i), id.) - d_{A_1,\dots,A_k}(\varphi^m(a_i), \varphi^m(\hat{a}_i)) \right| \leq c \ . \tag{3}$$

Let $(X, d_{A_1,\dots,A_k})$ denote the metric space which is the Cayley graph X of the ambient group F_n equipped with the metric $d_{A_1,\dots,A_k}$ defined above. F_n acts isometrically on $(X, d_{A_1,\dots,A_k})$ by left translations.
Let $f_1, \dots, f_n$ be a generating set for the free group F_n. If we set $\mu_m = \max_{1 \leq j \leq n} d_{A_1,\dots,A_k}(\varphi^m(f_j), id.)$ then by our assumptions, after possibly passing to a subsequence, $\mu_m \to \infty$.

Since restricted geodesics in the metric $d_{A_1,\dots,A_k}$ are quasi-geodesics in the word metric by lemma 3.12, and since the rescaling factors $\mu_m \to \infty$, we can extract from the sequence of actions of F_n on the metric spaces $(X, d_{A_1,\dots,A_k})$ a subsequence converging to a real tree Y by theorem 1.1 ([Pa1],2.3). The action of F_n on Y is stable and stabilizers of segments are either trivial or maximal cyclic by the proof of proposition 1.2 ([Ri-Se2],4.1-4.2). Furthermore, by the existence of the constant c and the inequality (3), each of the periodic factors A_i is a subgroup of a point stabilizer in Y.

Since φ acts on the metric space $(X, d_{A_1,\dots,A_k})$ as a bi-Lipschitz equivariant map, and since $\mu_m \to \infty$, we can also construct a commutative diagram (1) from a converging subsequence of actions of F_n on the metric space $(X, d_{A_1,\dots,A_k})$ via powers of the automorphisms φ. Since φ was assumed to have no periodic conjugacy classes, the commutative diagram (1) implies that Y contains no IET components and that the stabilizer of every segment in Y is trivial. Hence, the stabilizer of every point in Y is a periodic factor with respect to φ. Furthermore, by theorem 1.3, if $y_1, \ldots, y_r$ are points from distinct orbits in Y and with non-trivial stabilizers $\hat{P}_1, \ldots, \hat{P}_r$, then there exist conjugates $P_1, \ldots, P_r$ of $\hat{P}_1, \ldots, \hat{P}_r$ so that $P_1 * \ldots * P_r$ is a free factor in F_n. Since every periodic factor A_i can be conjugated into one of the point stabilizers P_j the proposition follows.

$\square$

By lemma 3.4 and under the assumptions of theorem 3.11, if $A_1, \ldots, A_k$ generate F_n then $F_n = A_1 * \ldots * A_k$. By proposition 3.13 if the sequence $\{d_{A_1,\ldots,A_k}(\varphi^m(f), id.)\}$ remains bounded for every $f \in F_n$, there exists $i > 1$ and a conjugate $\hat{A}_i$ of A_i so that $H = A_1 * \hat{A}_i$ $(i > 1)$ is invariant under φ^{n_0} for some power n_o. By theorem 3.5 H is a periodic factor with respect to φ, and by lemma 3.4 and our assumptions H intersects trivially every conjugate of A_j for $j \neq 1, i$. Hence, in the bounded case, proposition 3.13 allows us to reduce the number of periodic factors in the claim of the theorem.
If there exists an element $f \in F_n$ for which the sequence $\{d_{A_1,\ldots,A_k}(\varphi^m(f), id.)\}$ is not bounded, proposition 3.14 reduces the claim of the theorem to an ambient group with lower rank. Therefore, a finite induction on the number of periodic factors $A_1, \ldots, A_k$ and the rank of the ambient group F_n concludes the proof of theorem 3.11.

$\square$

4. Irreducible Extensions of Periodic Subfactors.

In section 2 we introduced the basic properties of periodic factors and their Dehn and irreducible extensions. In section 3 we studied the basic algebraic connections between distinct periodic factors which are sufficient to define the basic level in our hierarchical decomposition for automorphisms with no periodic conjugacy classes. To climb to higher levels in the hierarchical decomposition it is necessary to have a better understanding of the connections between general periodic factors, which, in particular, requires a closer look at irreducible extensions. This is our goal in this section.

The extensions defined in definition 2.3 are extensions of a unique periodic factor. When studying irreducible extensions a generalization to an extension of a finite number of (distinct) periodic factors is required. Recall that throughout this paper we assume that the automorphisms is question admit no (non-trivial) periodic conjugacy classes.

Definition 4.1 *Let φ be an automorphism of a free group F_n, and let $P_1, \ldots, P_m$ be $m > 1$ periodic free factors with respect to φ. Let A be a periodic free factor preserved by φ, and suppose $A = P_1 * \ldots * P_m * B$ for some (non-trivial) free factor $B < A$. A is called an irreducible extension of the periodic factors $P_1, \ldots, P_m$ if every periodic factor that is properly contained in A can be conjugated into one of the periodic factors $P_1, \ldots, P_m$.*

Irreducible extensions of a finite number of periodic free factors have similar properties to an irreducible extension of a unique periodic factor. Let φ be an automorphism of a free group F_n, and let A be an irreducible extension of the periodic factors $P_1, \ldots, P_m$ for some $m \geq 1$. Since A is preserved by φ, and $P_1, \ldots, P_m$ are periodic free factors, there exists a power k so that a composition of φ^k with an inner automorphism preserves the periodic factors A and P_1 and the conjugacy classes of the periodic factors $P_2, \ldots, P_m$. Let ν be the composition of φ^k with that inner automorphism.

Proposition 4.2 *With the notation and assumptions above let $A = P_1 * \ldots * P_m * B$ be an irreducible extension of the periodic factors $P_1, \ldots, P_m$, and let Δ be a graph of groups with fundamental group A, m vertices with vertex groups $P_1, \ldots, P_m$ and bouquet of circles corresponding to a set of free generators of B. Let T be the Bass-Serre tree corresponding to Δ, let t_1 be the vertex stabilized by P_1 in T, and let $A = < a_1, \ldots, a_s >$. Then:*

- (i) *for every constant λ, there exists an r_0 for which for all $r > r_0$ there exists an index $1 \leq i \leq s$ so that: $d_T(t_1, \nu^r(a_i)(t_1)) \geq \lambda$.*
- (ii) *Every limit tree obtained from a convergent subsequence of actions of A on the real tree T via the automorphisms ν^r is not simplicial.*

Proof: If the sequence of distances $\{d_T(t_1, \nu^r(b_i)(t_1))\}$ contains a bounded subsequence, then by the argument used to prove proposition 3.13 for some $i \leq m$, some conjugate $\hat{P}_i$ of P_i and some power n_0 of ν, $P_1 * \hat{P}_i$ is invariant under ν^{n_0}, and by theorem 3.5 $P_1 * \hat{P}_i$ is a periodic factor in A. Since A is assumed an irreducible extension this implies that m is necessarily 1.
If $m = 1$ and the sequence of distances $\{d_T(t_1, \nu^r(b_i)(t_1))\}$ contains a bounded subsequence, then by the argument used to prove lemma 3.8 A contains a Dehn ex-

tension of P_1 which contradicts the assumption of A being an irreducible extension and we get (i).

To prove (ii) note that if there exists a convergent subsequence of actions of A on the real tree T via the automorphisms ν^r with a simplicial limit, we can construct a commutative diagram (1) for the automorphism ν with both limit trees Y and Y^1 being simplicial.
Let Λ be the Bass-Serre graph of groups corresponding to the action of B on the (simplicial) limit tree Y. Since $P_1, \ldots, P_m$ fix points in T, and since every periodic factor properly contained in A can be conjugated into $P_1, \ldots, P_m$, conjugates of $P_1, \ldots, P_m$ must be the only non-trivial vertex groups in Λ.

If Λ contains a simple loop in which all vertex groups are trivial, then by the commutative diagram (1) the conjugacy class of the Bass-Serre generator corresponding to that loop is periodic, a contradiction to our assumption on A being a periodic factor, and in particular having no periodic conjugacy classes. If Λ contains a simple loop where only a single vertex has non-trivial stabilizer, then by the commutative diagram (1) the free factor generated by the single non-trivial vertex group and the Bass-Serre generator corresponding to the simple loop is periodic, and this periodic factor, which is properly contained in A, can not be conjugated into any of the periodic factors $P_1, \ldots, P_m$ which contradicts our assumptions on A being an irreducible extension. If Λ contains a simple loop with at least two non-trivial vertex groups, then by the commutative diagram (1), the free factor, which is the free product of two consecutive non-trivial vertex group in the simple loop, is a periodic factor, and since it is properly contained in A and it can not be conjugated into one of the periodic factors $P_1, \ldots, P_m$ we have obtained a contradiction to A being an irreducible extension of the P_i's.

□

If P_1 and P_2 are periodic free factors contained in a periodic factor B, and P_1 intersects trivially every conjugate of P_2, lemma 2.8 shows that $GD_B(P_1)$ intersects trivially every conjugate of $GD_B(P_2)$. The rest of this section shows that analogous statements hold for irreducible extensions.

Lemma 4.3 *Let φ be an automorphism of a free group F_n, let B be a periodic factor preserved by φ, and let $P_1, \ldots, P_m < B$ be periodic free factors with respect to φ so that P_i intersects trivially every conjugate of $P_{i'}$ for $i \neq i'$. Let $I < B$ be an irreducible extension of periodic factors $A_{i,j} = I \cap b_j P_i b_j^{-1}$ for some $b_j \in B$, and suppose I is preserved by φ. Then $I \cap bGD_B(P_i)b^{-1}$, where $b \in B$, is either trivial or it is a conjugate of one of the periodic factors $A_{i,j}$.*

Proof: By lemmas 2.6 and 2.7 it is enough to show that $I \cap bD_B(P_i)b^{-1}$, for some $b \in B$, is either trivial or a conjugate of one of the periodic factors $A_{i,j}$. If $L = I \cap bD_B(P_i)b^{-1}$ properly contains a conjugate of one of the periodic factors $A_{i',j'}$, then by the irreducibility of the extension I and our condition on the triviality of intersections between conjugates of the P_i's $L = I$. Hence, by lemma 2.6, I is a Dehn extension of $A_{i',j'}$, a contradiction. Since the periodic factor P_i intersects trivially every conjugate of the periodic factor $P_{i'}$ for every $i' \neq i$, necessarily $i = i'$ and $A_{i',j'} = A_{i,j}$ for some j.

□

A similar connection between two irreducible extensions follows immediately from its definition.

Lemma 4.4 *With the notation and assumptions of lemma 4.3 let $I, J < B$ be non-conjugate irreducible extensions of the periodic factors $M_{i,j} = I \cap g_j P_i g_j^{-1}$ and $N_{i,j} = J \cap f_j P_i f_j^{-1}$ in correspondence, where some of the $M_{i,j}$'s and $N_{i,j}$'s may be trivial. Then the intersection of a conjugate of J with I is either trivial or it can be conjugated into one of the periodic factors $M_{i,j}$.*

The following lemma is another basic property of irreducible extensions which plays a role in climbing through the levels of our hierarchical decomposition.

Lemma 4.5 *Let φ be an automorphism of a free group F_n, and let $P_1, \dots, P_r$ and $Q_1, \dots, Q_s$ be periodic free factors with respect to an automorphism φ that admits no periodic conjugacy classes. Suppose that P_i intersects trivially every conjugate of $P_{i'}$ for $i' \neq i$, Q_j intersects trivially every conjugate of $Q_{j'}$ for $j' \neq j$, and that $Q = Q_1 * \dots * Q_s$ is a periodic factor preserved by φ.*
If A is an irreducible extension of the periodic factors $P_1, \dots, P_r$ ($A = P_1 * \dots * P_r * B$ for some (non-trivial) free factor $B < A$) and A can not be conjugated into any of the periodic factors $Q_1, \dots, Q_s$, then A can not be conjugated into $Q = Q_1 * \dots * Q_s$.

Proof: Since $Q = Q_1 * \dots * Q_s$ is a periodic factor preserved by φ, basic properties of automorphisms of free products ([F-R1],[F-R2]) (and alternatively the argument used to prove proposition 3.13) imply that by possibly raising φ to a power and composing it with an appropriate inner automorphism of Q, rearranging the periodic factors $Q_1, \dots, Q_s$ and replacing them by their (appropriately chosen) conjugates, we may assume that $Q_1 * Q_2$ is a periodic factor with respect to φ, and (still) $Q = Q_1 * \dots * Q_s$. Hence, a finite reduction implies that to prove the lemma we may assume that $s = 2$, i.e., we may assume that $Q = Q_1 * Q_2$, the automorphism φ preserves the periodic factor Q_1, and $\varphi(Q_2) = q_1 Q_2 q_1^{-1}$ for some $q_1 \in Q_1$.
Suppose the irreducible extension A can be conjugated into the periodic factor Q. By replacing A with its conjugate we may assume that $A < Q$, and by possibly raising φ to a higher power we may also assume that φ preserves the conjugacy class of A.

Let Δ be the graph of groups with fundamental group Q, 2 vertices stabilized by Q_1 and Q_2, and an edge with trivial stabilizer connecting between the two vertices. Let T be the Bass-Serre tree corresponding to Δ, and let $t_1 \in T$ be the point stabilized by Q_1. Since Q_1 is invariant under φ and $\varphi(Q_2) = q_1 Q_2 q_1^{-1}$ for some $q_1 \in Q_1$, φ preserves distances on T, i.e., for every $q \in Q$: $d_T(\varphi(q)(t_0), t_0) = d_T(q(t_0), t_0)$. Therefore, we can construct a commutative diagram (2) from the action of Q on T.

Since $A < Q$, and since we assume A can not be conjugated into neither Q_1 nor Q_2, A inherits a (non-trivial) graph of groups Λ_A from its action on T. Let V_A be a maximal tree of groups in Λ_A, and let $U_1, \dots, U_z$ be the stabilizers of vertices in V. $U_1, \dots, U_z$ are periodic factors, each can be conjugated into one of the periodic factors Q_1 or Q_2, and since the conjugacy class of A is preserved by φ, the commutative diagram (2) corresponding to the action of Q on T implies (see the proof of lemma 2.6 above) that the fundamental group of every subtree of V_A is a periodic factor with respect to φ.
In particular, the fundamental group of the maximal tree V_A is a periodic factor in A. So if the graph of groups Λ_A is not a tree of groups, the commutative diagram (2) for the action of Q on T implies that A is a Dehn extension of the fundamental group of V_A. Since A is assumed an irreducible extension, it is not a Dehn extension

of a periodic factor properly contained in it, so Λ_A is a tree of groups.

Since every proper periodic factor in A can be conjugated into one of the periodic factors $P_1, \ldots, P_r$, each of the periodic factors $U_1, \ldots, U_z$ can be conjugated into one of the periodic factors $P_1, \ldots, P_r$. Since the fundamental group of every proper subtree of Λ_A is a periodic factor as well, if Λ_A contains more than 2 vertices $A = U_1 * \ldots * U_z$ can be conjugated into one of the periodic factors P_i, a contradiction since P_i is a proper subfactor in the irreducible extension A.
If V contains only 2 vertices, U_1 can be conjugated into P_1 and U_2 can be conjugated into P_2, then $rank(A) = rank(U_1) + rank(U_2) \leq rank(P_1) + rank(P_2)$, a contradiction to A being an irreducible extension. If both U_1 and U_2 can be conjugated into P_1 then $rank(A) = rank(U_1) + rank(U_2) \leq rank(P_1)$, again a contradiction to A being an irreducible extension.

□

Definition 4.1, lemmas 4.2-4.5 and the techniques presented in the previous section are the basic tools needed in handling irreducible extensions of subfactors. Throughout the rest of this section we will assume that A is an irreducible extension of the periodic factors $M_1, \ldots, M_s$ with respect to an automorphism φ, and that $A = B * M_1 * \ldots * M_s$ for some (non-trivial) free factor B. We assume that there exist periodic factors $P_1, \ldots, P_k$ in F_n where $k \leq s$, so that P_i intersects A in the periodic factor M_i for $i = 1, \ldots, k$, and P_i intersects trivially every conjugate of $P_{i'}$ for $i \neq i'$. For each $k+1 \leq j \leq s$ we assume that the periodic factor M_j is a subfactor of a periodic factor $g_j P_{i_j} g_j^{-1}$ for $j = k+1, \ldots, s$. To shorten notation we will assume that φ preserves A and P_1, that φ preserves the conjugacy classes of the periodic factors $P_2, \ldots, P_k$, and that $\varphi(g_j P_{i_j} g_j^{-1}) = u_j g_j P_{i_j} g_j^{-1} u_j^{-1}$ for $j = k+1, \ldots, s$ and some $u_j \in A$. Clearly, if the automorphism in question does not satisfy this last assumption, it will hold for composition of a power of it with an appropriate inner automorphism. with these notation and assumptions we call the subgroup $H = < A, P_1, \ldots, P_k, g_{k+1}, \ldots, g_s >$ an *irreducible brick* with respect to φ. This section is (mostly) devoted to the algebraic structure of an irreducible brick, where the main goal is to obtain that H is a periodic factor in the ambient group F_n with respect to the automorphism φ.

As we did in studying the algebraic connections between distinct periodic factors, we start by looking at the algebraic structure of the subgroup generated by the distinct periodic factors $P_1, \ldots, P_k$, the irreducible extension A, and the conjugating elements $g_{k+1}, \ldots, g_s$.

Lemma 4.6 *With the notation and assumptions above:*

$$H = < A, P_1, \ldots, P_k, g_{k+1}, \ldots, g_s > = B * P_1 * \ldots * P_k * < g_{k+1} > * \ldots * < g_s >$$

Proof: Since φ preserves the periodic factors A and P_1, and A is an irreducible extension of the periodic subfactors M_j for $j = 1, \ldots, s$, φ preserves the subgroup H. Since the periodic factor P_i intersects trivially every conjugate of the periodic factor $P_{i'}$ for $i \neq i'$, theorem 3.11 implies that there exist conjugates $\hat{P}_1, \ldots, \hat{P}_k$ of $P_1, \ldots, P_k$ so that $\hat{P}_1 * \ldots * \hat{P}_k$ is a free factor in the ambient group F_n. Suppose $F_n = \hat{P}_1 * \ldots * \hat{P}_k * < q_1 > * \ldots * < q_\ell >$, and assume (w.l.o.g.) that $P_1 = \hat{P}_1$.

Let Δ be a graph of groups with fundamental group F_n, k vertices stabilized by $\hat{P}_1, \ldots, \hat{P}_k$, $k-1$ edges connecting between the k vertices, and an additional bouquet of ℓ circles with corresponding Bass-Serre generators $q_1, \dots, q_\ell$ placed on the vertex stabilized by $P_1 = \hat{P}_1$. Let T be the Bass-Serre tree corresponding to Δ, and let $t_0 \in T$ be the vertex stabilized by P_1.

Since the subgroup H is invariant under the automorphism φ, from the sequence of actions of H on the pointed tree (T, t_0) it is possible to extract a subsequence converging into a real tree Y_H, and a corresponding commutative diagram (1) between the limit trees Y_H^1 and Y_H (see theorem 1.5 above). Since H contains no periodic conjugacy classes, the stabilizer of every non-degenerate segment in Y_H is trivial. Let Y_A be the minimal subtree of Y_H preserved by the periodic factor A.
By construction, every conjugate of P_i in H fixes a point in Y_H, and since Y_H is not a single point, and H is generated by the periodic factors A and $P_1, \dots, P_k$ and the elements $g_{k+1}, \dots, g_s$, Y_A is not a single point as well. Since the periodic factors $M_1, \dots, M_s$ are all contained in the periodic factor A, $M_j < P_j$ for $j = 1, \dots, k$ and $M_j < g_j P_{i_j} g_j^{-1}$ for $j = k+1, \dots, s$, the periodic factors P_i and $g_j P_{i_j} g_j^{-1}$ fix points in Y_A for $i = 1, \dots, k$ and $j = k+1, \dots, s$.

If A-conjugates of the periodic factors P_i and $P_{i'}$ for $i \neq i'$ fix the same point in Y_A, or A-conjugates of P_i and $g_j P_{i_j} g_j^{-1}$ fix the same point in Y_A, the commutative diagram (1) implies that there exists a periodic factor Q properly contained in A, and Q contains A-conjugates of the periodic factors M_j and $M_{j'}$ for some $j \neq j'$. Since the periodic factor A is an irreducible extension of the periodic factors $M_1, \dots, M_s$, every proper periodic factor in A can be conjugated into one of the M_j's. Hence, A-conjugates of the periodic factors P_i and A-conjugates of the periodic factors $g_j P_{i_j} g_j^{-1}$ have to fix distinct points in Y_A for $i = 1, \dots, k$ and $j = k+1, \dots, s$.

Let $j_1, \dots, j_r$ be all the indices for which $P_{i_j} = P_i$. Let $N_i < P_i$ be a free factor for which:

$$P_i = N_i * M_i * g_{j_1}^{-1} M_{j_1} g_{j_1} * \dots * g_{j_r}^{-1} M_{j_r} g_{j_r} \ .$$

Clearly, the free factors B, $N_1, \dots, N_k$, $M_1, \dots, M_s$ and the elements $g_{k+1}, \dots, g_s$ generate H.
Proposition 3.2 implies that if $h \in H$ then $h(Y_A)$ intersects Y_A in a non-degenerate segment if and only if $h \in A$. Hence, $g_j(Y_A)$ intersects Y_A in a single point in Y_H, and so do $p_i(Y_A)$ for $p_i \in P_i$; $p_i \notin A$ and $g_j p_{i_j} g_j^{-1}$ for $p_{i_j} \in P_{i_j}$; $g_j p_{i_j} g_j^{-1} \notin A$. Therefore, the tree Y_H is a union of copies of translates of Y_A, each two distinct translates intersect at most in one point, and the shortest path between any two points in Y_H is supported on finitely many copies of translates of Y_A.

This structure of Y_H allows us to define a natural distance function between copies of Y_A in Y_H. We define the distance from $h_1(Y_A)$ to $h_2(Y_A)$ to be the minimal number of copies of Y_A, excluding $h_1(Y_A)$, on which a path from a non-branching point in $h_1(Y_A)$ to a non-branching point in $h_2(Y_A)$ is supported. We denote by $\eta(h)$ the distance from Y_A to $h(Y_A)$. Clearly, if $h_1(Y_A)$ and $h_2(Y_A)$ intersect in a single point then $\eta(h_1) - \eta(h_2)$ is either -1,0 or 1. Since A-conjugates of P_i and $g_j P_{i_j} g_j^{-1}$ fix points in distinct A-orbits in Y_A, if w is a reduced word in $g_{k+1}, \dots, g_s$ and free bases of the free factors B, $N_1, \dots, N_k$, $M_1, \dots, M_s$, then η is a non-decreasing function on prefixes (as well as suffixes) of w. Since these elements generate H, and $\eta(w) = 0$ only when w is a word in the free bases of B and $M_1, \dots, M_s$ exclusively, and by our assumptions $A = B * M_1 * \dots * M_s$, it follows that:

$$H = B * M_1 * \dots * M_s * N_1 * \dots * N_k * < g_{k+1} > * \dots * < g_s >$$

and the lemma follows.

□

Lemma 4.6 studies the algebraic structure of H, the subgroup generated by the periodic factors A, $P_1, \dots, P_k$ and the elements $g_{k+1}, \dots, g_s$. Like in the previous section, the main goal of this section is showing that the subgroup H is a periodic factor in F_n.

Theorem 4.7 *With the notation and assumptions above the irreducible brick $H = < A, P_1, \dots, P_k, g_{k+1}, \dots, g_s >$ is a periodic factor with respect to the automorphism φ.*

Proof: The strategy of our proof is similar to the one used to prove theorems 3.5 and 3.11. By lemmas 2.8, 4.3 and 4.6 we may assume w.l.o.g. that the periodic factors $P_1, \dots, P_k$ admit no Dehn extensions.
If H, the subgroup generated by the periodic factors A and $P_1, \dots, P_k$, and the elements $g_{k+1}, \dots, g_s$, is the entire group F_n, the theorem is obvious, so suppose F_n is generated by A, $P_1, \dots, P_k$, and the elements $f_1, \dots, f_\alpha \in F_n$ where α is the minimal number of elements needed to be supplemented to H in order to generate F_n. Let G be a free group given by the free product $G = R * S_1 * \dots * S_k * < u_1 > * \dots * < u_\alpha >$ where $R \simeq A$, $S_i \simeq P_i$, and let $\rho : G \to F_n$ be the natural epimorphism sending R to A, S_i to P_i, and u_i to f_i for $i = 1, \dots, \alpha$.

We denote by $\ell_{A,P_1,\dots,P_k}(g)$ the length of a normal form of an element $g \in G$. This length function on G defines a natural metric on F_n through the epimorphism ρ:

$$d_{A,P_1,\dots,P_k}(f, id.) \ = \ \min_{g \in \rho^{-1}(f)} \ell_{A,P_1,\dots,P_k}(g)$$

Since A is invariant under φ, $d_{A,P_1,\dots,P_k}(\varphi^m(a), id.) = 1$ for $a \in A$. Similarly $d_{A,P_1,\dots,P_k}(\varphi^m(p), id.)$ is either 1 or 3 for $p \in P_i$, $i = 1, \dots, k$, and $d_{A,P_1,\dots,P_k}(\varphi^m(g_j), id.)$ remains bounded for $j = k+1, \dots, s$. For the rest of the proof of the theorem we fix a basis $a_1, \dots, a_q$ for the periodic factor A, and a basis $p_{i,1}, \dots, p_{i,q_i}$ for each of the periodic factors P_i ($i = 1, \dots, k$). To every word w in the a_i's, $p_{i,j}$'s, and f_i's we can associate a length $\ell_{A,P_1 \dots, P_k}(w)$ by naturally identifying it with a word in the group G.

For a word w in the a_i's, $p_{i,j}$'s and f_i's we denote by $\ell(w)$ the word length of w (i.e., the total number of a's, p's and f's in w). For such a word w let Q_w be the set of all words v in the a_i's, $p_{i,j}$'s and f_i's for which $v = w$ as an element of F_n and $\ell_{A,P_1,\dots,P_k}(v) = d_{A,P_1,\dots,P_k}(w, id.)$. We say that w is a *restricted geodesic* with respect to the metric $d_{A,P_1,\dots,P_k}$ if $w \in Q_w$ and $\ell(w) \ = \ \min_{v \in Q_w} \ell(v)$. Clearly, there are finitely many restricted geodesics connecting from the identity to a given vertex in the Cayley graph X of F_n. The argument used to prove lemma 3.6 naturally generalizes to show restricted geodesics are quasi-geodesics with respect to the word metric on the Cayley graph X.

Lemma 4.8 *With the notation and assumptions above, there exists a constant λ (depending only on $A, P_1, \dots, P_k$ their chosen bases and the elements f_i) so that if w is a restricted geodesic with respect to the metric $d_{A,P_1,\dots,P_k}$, then w is a λ-quasi-geodesic in the Cayley graph X of F_n equipped with the word metric.*

Like the proofs of theorems 3.5 and 3.11, to prove theorem 4.7 we need to separate our argument. We first study the case in which the sequence $\{d_{A,P_1,\dots,P_k}(\varphi^m(f), id.)\}$

is bounded for every (fixed) $f \in F_n$, and then treat the case in which it is not bounded.

Lemma 4.9 *With the notation and assumptions of theorem 4.7 suppose there exists a constant c so that $d_{A,P_1,\dots,P_k}(\varphi^m(f_i), id.) < c$ for every m and every $i = 1,\dots,\alpha$. Then there exists a power z_0 and elements $v_1,\dots,v_r \in F_n$ so that $v_j \notin H$, H and $v_1,\dots,v_r$ generate F_n, $\varphi^{z_0}(P_i) = u_i P_i u_i^{-1}$ where $u_i \in A$ $(i = 1,\dots,k)$, and for every $j = 1,\dots,r$ there exist non-trivial elements $p_j^1 \in P_{i_1(j)}$, $p_j^2 \in P_{i_2(j)}$ for some $1 \leq i_1(j) < i_2(j) \leq k$, and elements $a_j^1, a_j^2 \in A$, so that $\varphi^{z_0}(v_j)$ has one of the following forms:*

(i) $\varphi^{z_0}(v_j) = u_{i_1(j)} p_j^1 v_j p_j^2 u_{i_2(j)}^{-1}$

(ii) $\varphi^{z_0}(v_j) = a_j^1 v_j p_j^2 u_{i_2(j)}^{-1}$ *where $a_j^1 \notin P_i$ for $i = 1,\dots,s$.*

(iii) $\varphi^{z_0}(v_j) = a_j^1 v_j a_j^2$ *where a_j^1 can be trivial.*

Proof: Since $d_{A,P_1,\dots,P_k}(\varphi^m(f_i), id.)$ remains bounded for all powers m and $i = 1,\dots,s$, a pigeon-hole argument, similar to the one used in the proofs of lemma 3.8 and proposition 3.13, shows that there exists a fixed power n_0 of φ so that we can restrict to a subsequence of powers of φ for which the following words $w_m = \varphi^{n_m}(f_1)$ and $\hat{w}_m = \varphi^{n_m+n_0}(f_1)$, which are restricted geodesics with respect to the metric $d_{A,P_1,\dots,P_k}$, have the form:

$$\varphi^{n_m}(f_1) = w_m = x_{1,1}x_{1,2}\dots x_{1,i_1}t_1x_{2,1}\dots t_2\dots x_{e,i_e}t_e$$

$$\varphi^{n_m+n_0}(f_1) = \hat{w}_m = \hat{x}_{1,1}\hat{x}_{1,2}\dots \hat{x}_{1,i_1}t_1\hat{x}_{2,1}\dots t_2\dots \hat{x}_{e,i_e}t_e$$

where the subwords t_j are not trivial and independent of m, the subwords $x_{i,j}$ belong to one of the periodic factors A or $P_1,\dots,P_k$ and the length of the $x_{i,j}$ grows to ∞ in the word metric on F_n (see the proof of lemma 3.8 for a more detailed explanation of our assertions).

By lemma 4.8 the words w_m and $\hat{w}_m$ are λ-quasi-geodesics in the standard word metric on the Cayley graph X for some fixed constant λ. Since w_m is a λ-quasi-geodesic and the automorphism φ acts as a bi-Lipschitz equivariant map on the Cayley graph X equipped with the word metric, the words $w'_m = \varphi^{n_m+n_0}(f_1)$ given by:

$$\varphi^{n_m+n_0}(f_1) = w'_m = \varphi^{n_0}(x_{1,1})\varphi^{n_0}(x_{1,2})\dots\varphi^{n_0}(t_1)\varphi^{n_0}(x_{2,1})\dots\varphi^{n_0}(t_2)\dots\varphi^{n_0}(x_{e,i_e})\varphi^{n_0}(t_e)$$

are λ'-quasi-geodesics for some fixed constant λ'.

Since A and H are preserved by φ, $\varphi^{n_0}(P_i) = u_i P_i u_i^{-1}$, $\hat{w}_m = w'_m = \varphi^{n_m+n_0}(f_1)$, $\hat{w}_m$ is a λ-quasi-geodesic and w'_m is a λ'-quasi-geodesic, the lengths of the subwords $x_{i,j}$ grows to ∞, the periodic factors A and $P_1,\dots,P_k$ are malnormal in F_n, every conjugate of P_i intersects P_j trivially for $i \neq j$, and the words $\hat{w}_m$ are restricted geodesics with respect to the metric $d_{A,P_1,\dots,P_k}$, if $t_j \notin H$ then $\varphi^{n_0}(t_j)$ must have one of the following forms depending on whether the parts of the subwords of w_m before and after t_j belong to A or $P_1,\dots,P_k$ in correspondence:

(i) if x_{j,i_j}^m and $x_{j+1,1}^m$ are elements of A then $\varphi^{n_0}(t_j) = a_j^1 t_j a_j^2$ for some $a_j^1, a_j^2 \in A$. Either a_j^1 or a_j^2 must be non-trivial.

(ii) if x_{j,i_j}^m is an element of $P_{i_1(j)}$ and $x_{j+1,1}^m$ is an element of $P_{i_2(j)}$ then $\varphi^{n_0}(t_j) = u_{i_1(j)} p_j^1 t_j p_j^2 u_{i_2(j)}^{-1}$ for some $p_j^1 \in P_{i_1(j)}$ and $p_j^2 \in P_{i_2(j)}$. Since we assumed the periodic factors admit no Dehn extensions, $i_1(j) \neq i_2(j)$.

(iii) if x^m_{j,i_j} is an element of A and $x^m_{j+1,1}$ is an element of $P_{i_2(j)}$ then $\varphi^{n_0}(t_j) = a^1_j t_j p^2_j u^{-1}_{i_2(j)}$ for some $a^1_j \in A$ and $p^2_j \in P_{i_2(j)}$.

(iv) if x^m_{j,i_j} is an element of $P_{i_1(j)}$ and $x^m_{j+1,1}$ is an element of A then $\varphi^{n_0}(t_j) = u_{i_1(j)} p^1_j t_j a^2_j$ for some $p^1_j \in P_{i_1(j)}$ and $a^2_j \in A$.

By possibly replacing t_j with t_j^{-1}, we may identify cases (iii) and (iv), and assume $i_1(j) < i_2(j)$ in case (ii). Repeating the entire argument for $f_2, \dots, f_s$ and replacing the t_j's obtained for all the f_i's with $v_1, \dots, v_r$ we have $F_n =< H, v_1, \dots, v_r >$ and the lemma follows.

□

Like in the proof of theorem 3.5, lemma 4.9 allows us to complete the proof of theorem 4.7 in case the distance function $d_{A,P_1,\dots,P_k}$ remains bounded. Proposition 4.10 below concludes that in this bounded case the ambient group F_n is a Dehn extension of the periodic factor $H =< A, P_1, \dots, P_k, g_{k+1}, \dots, g_s >$.

Proposition 4.10 *With the notation and assumptions above suppose there exists a constant c so that $d_{A,P_1,\dots,P_k}(\varphi^m(f_i), id.) < c$ for every m and every $i = 1, \dots, s$. Then the irreducible brick $H =< A, P_1, \dots, P_k, g_{k+1}, \dots, g_s >$ is a periodic factor and the ambient group F_n is a Dehn extension of H with respect to the given automorphism φ.*

Proof: To prove the proposition one can either use the dynamics of the action of F_n on a limit real tree or give a combinatorial argument. Unlike proposition 3.9 we prefer to use the structure of the real tree in the limit.

Like in the construction we used in proving lemma 4.6, since the periodic factor P_i intersects trivially every conjugate of the periodic factor P_j for $i \neq j$, theorem 3.11 implies that there exist conjugates $\hat{P}_1, \dots, \hat{P}_k$ of $P_1, \dots, P_k$ so that $\hat{P}_1 * \dots * \hat{P}_k$ is a free factor of the ambient group F_n. Suppose $F_n = \hat{P}_1 * \dots * \hat{P}_k * < q_1 > * \dots * < q_\ell >$, and assume (w.l.o.g.) that $P_1 = \hat{P}_1$.

Let Δ be a graph of groups with fundamental group F_n, k vertices stabilized by $\hat{P}_1, \dots, \hat{P}_k$, $k-1$ edges connecting between the k vertices, and an additional bouquet of ℓ circles with corresponding Bass-Serre generators $q_1, \dots, q_\ell$ placed on the vertex stabilized by $P_1 = \hat{P}_1$. Let T be the Bass-Serre tree corresponding to Δ, and let $t_0 \in T$ be the vertex stabilized by P_1.

¿From the sequence of actions of F_n on the pointed tree (T, t_0) it is possible to extract a subsequence converging into a real tree Y, and a corresponding commutative diagram (1) between the limit trees Y^1 and Y by theorem 1.5. Since there are no periodic conjugacy classes with respect to φ, the stabilizer of every non-degenerate segment in Y and Y^1 is trivial. Let Y_A and Y_H be the minimal subtrees of Y preserved by the periodic factors A and H in correspondence.
By construction, every conjugate of P_i fixes a point in Y and Y^1. If Y_A is a single point in Y, Y_H is a single point as well, so $H < R$ where R is a point stabilizer in Y, and by the commutative diagram (1) R is a proper periodic factor in F_n preserved by φ. Hence, the case of Y_A being a single point follows by induction on the rank of the ambient group, and for the rest of the argument we assume that Y_A is non-degenerate.

Since A is an irreducible extension, proposition 3.1 implies that A admits a dense orbit when acting on Y_A, H when acting on Y_H, and F_n when acting on Y. Since the periodic factors $M_1, \dots, M_s$ are all contained in the periodic factors A and

$M_i < P_i$ for $i = 1, \dots, k$ and $M_j < g_j P_{i_j} g_j^{-1}$ for $j = k+1, \dots, s$, the periodic factors P_i and $g_j P_{i_j} g_j^{-1}$ fix points in Y_A for $i = 1, \dots, k$ and $j = k+1, \dots, s$. Since the periodic factor A is an irreducible extension of the periodic factors $M_1, \dots, M_s$, A-conjugates of P_i and A-conjugates of P_j or $g_j P_{i_j} g_j^{-1}$ fix distinct points in Y_A for $i \neq j$.

Let $y_1, \dots, y_k$ be points stabilized by $P_1, \dots, P_k$ in Y_A, and suppose $y_1, \dots, y_k$ belong to only k_1 distinct F_n-orbits. Let $R_1, \dots, R_{k_1}$ be stabilizers of points $y_{i_1}, \dots, y_{i_{k_1}}$ which belong to all the distinct F_n-orbits in Y. Clearly, the union of $R_1, \dots, R_{k_1}$ contains conjugates of all the periodic factors $P_1, \dots, P_k$, w.l.o.g. we may assume that $A \cap R_i = M_i$ for $i = 1, \dots, k_1$, and for some elements $w_j \in F_n$, $A \cap w_j R_{i_j} w_j^{-1} = M_j$ for $j = k_1 + 1, \dots, s$.
Let $H_1 =< A, R_1, \dots, R_{k_1}, w_{k_1+1}, \dots, w_s >$. φ preserves H_1, by construction $H < H_1$, and by the algebraic structure of H and H_1 given by lemma 4.6, H is a periodic factor preserved by φ in H_1. Hence, if H_1 is a periodic factor in F_n so is H, so proposition 4.10 stated for the subgroup H follows from the same proposition stated for H_1 joint with lemma 4.9 for the subgroup H itself. Since, by construction, H_1 contains a strictly smaller number of periodic factors if $k_1 < k$, proposition 4.10 follows by induction on the number of periodic factors in this case, so for the rest of the proof of proposition 4.10 we may assume that conjugates of the periodic factors P_i and $P_{i'}$ stabilize distinct points in Y for $i \neq i'$. The next lemma classifies the stabilizers of points in Y in this case.

lemma 4.11 *With the notation above and the assumptions of proposition 4.10 suppose conjugates of P_i and $P_{i'}$ stabilize distinct points if $i \neq i'$. Then the stabilizer of a point in Y_A (and Y_H) is a conjugate of one of the periodic factors $P_1, \dots, P_k$.*

Proof: Let D be a (non-trivial) stabilizer of a point in Y_A. Since A is preserved by φ, the commutative diagram (1) and the finiteness of orbits of germs of edges issuing from branching points in Y_A (theorem 1.10) imply that for some power ℓ_0 of φ and some $a'_{\ell_0} \in A$: $\varphi^{\ell_0}(D) = a'_{\ell_0} D {a'_{\ell_0}}^{-1}$. After possibly composing φ^{ℓ_0} with an inner automorphism we may assume that φ^{ℓ_0} preserves both D and H.

Suppose first that D contains no conjugate of P_i for every $i = 1, \dots, k$. Then, by the structure theory for stable actions of groups on real trees (theorem 1.3 above) and the commutative diagram (1), D is a periodic factor of F_n that intersects trivially every conjugate of the periodic factors A and P_i, and D is in particular a malnormal quasi-convex subgroup of F_n.
Since D is invariant under φ^{ℓ_0}, since restricted geodesics with respect to the metric $d_{A,P_1,\dots,P_k}$ are quasi-geodesics with respect to the word metric on F_n by lemma 4.8, and since D is malnormal quasi-convex subgroup of F_n that intersects every conjugate of A and the P_i's trivially, a bound on $d_{A,P_1,\dots,P_k}(\varphi^{m\ell_0}(d), id.)$ which is independent of the power m, implies a bound on $\varphi^{m\ell_0}(d)$ in the word metric on D for every fixed $d \in D$. Hence, if the point stabilizer D contains no conjugate of the P_i's then D contains periodic conjugacy classes with respect to φ which contradicts our assumptions.

To conclude the proof of the lemma suppose D contains a conjugate of one of the periodic factors P_i, w.l.o.g. $P_1 < D$. In this case, D is a periodic factor of F_n, P_1 is a periodic factor of D, φ^{ℓ_0} preserves both D and P_1, and D intersects trivially every conjugate of $P_2, \dots, P_k$ and intersects A in M_1. D is clearly malnormal and quasi-convex in F_n.

Let $D = P_1 * < d_1 > * \ldots * < d_g >$, let T_D be the Bass-Serre tree corresponding to this free product, and let $t_1 \in T_D$ be the vertex stabilized by P_1. By the same argument given above, a bound on $d_{A,P_1,\dots,P_k}(\varphi^{m\ell_0}(d), id.)$ which is independent of the power m, implies a bound on $d_{T_D}(\varphi^{m\ell_0}(d)(t_1), t_1)$ in the word metric on D. Hence, in case $P_1 < D$, D is a Dehn extension of P_1. Since the periodic factors P_i were assumed to have no Dehn extensions, $D = P_1$.

□

Studying the stabilizers of points in Y_A, to prove proposition 410. we need to get a better understanding of the structure of the global limit tree Y, and the action of the elements $v_1, \dots, v_r$ (defined in lemma 4.9) on it.

Lemma 4.12 *With the notation and assumptions above, $v_j(Y_H)$ intersects Y_H in a single point for $j = 1, \dots, r$.*

Proof: Let $G_j =< v_j, H >$. By lemma 4.9 G_j is clearly a φ^{z_0}-invariant f.g. subgroup of F_n, and H is a proper subgroup of G_j. Since Y_H contains a dense orbit with respect to the action of H, and H is a subgroup of G_j, Y_j, the minimal subtree preserved by G_j in Y, contains a dense orbit with respect to the action of G_j according to proposition 3.1. Let Z_j be the minimal subtree of Y containing both Y_H and $v_j(Y_H)$, $Z_j = Y_H \cup I \cup v_j(Y_H)$. Our goal is to show that the interval I is degenerate.

Suppose the interval I is non-degenerate. If $v_j^{-1}(I) \cap Y_H$ and $I \cap Y_H$ belong to different H-orbits in Y_H, the interior of $g(I)$ does not intersect I for every non-trivial $g \in G_j$. Hence, the action of G_j on Y_j contains a discrete part, a contradiction. If for some $h \in H$, $hv_j^{-1}(I) \cap I$ contains the middle point of I, this point is stabilized by hv_j^{-1}, and since the stabilizer of every non-degenerate segment in Y is trivial the middle point of I is the only point stabilized by hv_j^{-1}. Since G_j is generated by H and hv_j^{-1}, the action of G_j on Y_j contains discrete parts in this case as well, a contradiction. If for some $h \in H$, $hv_j^{-1}(I) \cap I$ is non-trivial but does not contain the middle point of I, hv_j^{-1} is a hyperbolic element when acting on Y_j, and the axis of hv_j^{-1} is mapped to itself only by powers of itself. Hence, Y_j contains discrete parts in this last case as well, a contradiction, and the lemma follows.

□

By lemma 4.11 the assumptions of proposition 4.10 and assuming conjugates of P_i and $P_{i'}$ stabilize distinct points if $i \neq i'$ (which we can always assume by induction on the number of periodic factors $P_1, \dots, P_k$), the stabilizer of a point in Y_A (and Y_H) is a conjugate of one of the periodic factors $P_1, \dots, P_k$. Since the action of the irreducible extension A on its minimal subtree Y_A has a dense orbit, and since there are only finitely many orbits of germs of edges issuing from a branching point in Y_A by theorem 1.10, there are only finitely many A-orbits of points in Y_A with non-trivial stabilizers.

Let $y_1, \dots, y_q \in Y_A$ be points from distinct A-orbits so that $y_1, \dots, y_q$ represent all the distinct orbits of points with non-trivial stabilizer in Y_A. Since A is an irreducible extension of the periodic subfactors $M_1, \dots, M_s$, $P_1, \dots, P_k$ and $g_{k+1} P_{i_{k+1}} g_{k+1}^{-1}, \dots, g_s P_{i_s} g_s^{-1}$ stabilize points from distinct A-orbits in Y_A. Hence we may suppose that $y_1, \dots, y_s$ are stabilized by these last periodic factors. Since each of the y_j is stabilized by a conjugate of one of the periodic factors P_i by lemma 4.11, let $t_j P_{i_j} t_j^{-1}$ stabilize the points $y_{s+1}, \dots, y_q$.

If L is the subgroup generated by H and the elements $t_{s+1}, \dots, t_q$ then L is preserved by φ, L is a malnormal subgroup, and by the proof of lemma 4.6:

$$L = < H, t_{s+1}, \dots, t_q > = H* < t_{s+1} > * \dots * < t_q >$$

Therefore, to prove that H is a periodic factor with respect to φ, it is enough to show that L is a periodic factor with respect to φ.

Like the argument used in proving lemma 4.6, lemma 4.12 and proposition 4.9 allow us to define a natural distance function between copies of Y_L in Y. We define the distance from $g_1(Y_L)$ to $g_2(Y_L)$ to be the minimal number of copies of Y_L, excluding $g_1(Y_L)$, on which a path from a non-branching point in $g_1(Y_L)$ to a non-branching point in $g_2(Y_L)$ is supported. We denote by $\eta(g)$ the distance from Y_L to $g(Y_L)$. Clearly, if $g_1(Y_L)$ and $g_2(Y_L)$ intersect in a single point then $\eta(g_1) - \eta(g_2)$ is either -1,0 or 1.

Let $e_1, \dots, e_\beta$ be a minimal subset of the elements $v_1, \dots, v_r$ given by proposition 4.9, so that L and $e_1, \dots, e_\beta$ generate F_n. Our goal in proving proposition 4.5 is showing that $F_n = L* < e_1 > * \dots * < e_\beta >$. If this last assertion does not hold there must exist a non-trivial relation between L and the e_i's, so let $w = e_{i_1}^{\pm 1} \ell_1 e_{i_2}^{\pm 1} \ell_2 \dots e_{i_k}^{\pm 1} \ell_m$ where $\ell_i \in L$ be such a relation that contains the minimal appearances of the elements $e_1, \dots, e_\beta$. If one of the e_i's appears exactly once in w this e_i can be expressed as a word in L and the other e_j's which contradicts the minimality of the set $e_1, \dots, e_\beta$. Since L is malnormal $e_i \ell e_i^{-1} \notin L$ for every non-trivial $\ell \in L$.

Let w_j be the j-prefix of the word w, i.e., $w_j = e_{i_1}^{\pm 1} \ell_1 \dots e_{i_j}^{\pm 1} \ell_j$. Since $\eta(w_1) = 1$, $\eta(w) = \eta(w_m) = 0$, and the difference between $\eta(w_j)$ and $\eta(w_{j+1})$ is either 0,1 or -1, there must exist an index $2 \leq j \leq m$ for which $\eta(w_{j-1}) > \eta(w_j)$. Let j_0 be the minimal index for which the last inequality holds.

If $j_0 < m$ let $j_1 < j_0$ be the maximal index for which $\eta(w_{j_1}) = \eta(w_{j_0})$. By lemma 4.7 necessarily $w_{j_1}(Y_L) = w_{j_0}(Y_L)$, so $w_{j_0} = w_{j_1} \ell$ for some $\ell \in L$, a contradiction to w being a relation with minimal appearances of the elements e_i. Hence, $\eta(w_j) = 1$ for $j = 1, \dots, m-1$.

Since no e_i appears in w exactly once, there must exist an index $2 \leq j \leq k$ for which $e_{i_j} = e_{i_1}$. Let j_2 be the minimal such index. If $w_{j_2} = e_{i_1} \ell_1 \dots e_{i_{j_2}} \ell_{j_2}$ or $w_{j_2} = e_{i_1}^{-1} \ell_1 \dots e_{i_{j_2}}^{-1} \ell_{j_2}$, then since $\eta(w_j) = 1$ for every $1 \leq j \leq m$, $w_{j_2 - 1}$ fixes the point $Y_L \cap w_1(Y_L)$, so $w_{j_2-1} \in L$, a contradiction to w being a shortest relation.

If $w_{j_2} = e_{i_1} \ell_1 \dots e_{i_{j_2}}^{-1} \ell_{j_2}$ or $w_{j_2} = e_{i_1}^{-1} \ell_1 \dots e_{i_{j_2}} \ell_{j_2}$, then since $\eta(w_j) = 1$ for every $1 \leq j \leq m-1$, $\ell_1 e_{i_2} \dots e_{i_{j_2-1}} \ell_{j_2-1}$ fixes the point $Y_L \cap w_1(Y_L)$, so $\ell_1 e_{i_2} \dots e_{i_{j_2-1}} \ell_{j_2-1} \in L$, a contradiction to w being a shortest relation.

This finally shows that $F_n = L* < e_1 > * \dots * < e_\beta >$, hence L is a periodic factor with respect to φ and so is H since $L = H * t_{s+1} * \dots * t_q$. Since H is a periodic factor preserved by φ, lemma 4.9 implies that F_n is a Dehn extension of H with respect to φ.

□

To complete the proof of theorem 4.7 we need to study the case in which the distance function $d_{A,P_1,\dots,P_k}$ is unbounded. Like in the proofs of theorems 3.5 and 3.11 (propositions 3.10 and 3.14) to analyze this case we construct a commutative diagram from the actions of F_n on its Cayley graph X equipped with the metric $d_{A,P_1,\dots,P_k}$.

Proposition 4.13 *With the notation and assumptions above suppose there exists an element $f \in F_n$ for which $d_{A,P_1,\dots,P_k}(\varphi^m(f), id.)$ is not bounded. Then there exists a periodic factor R properly contained in F_n, R is preserved by φ and $< A, P_1, \dots, P_k, g_{k+1}, \dots, g_s >= H < R$.*

Proof: Let $(X, d_{A,P_1,\dots,P_k})$ denote the metric space which is the Cayley graph X of the ambient group F_n equipped with the metric $d_{A,P_1,\dots,P_k}$ defined above. F_n acts isometrically on $(X, d_{A,P_1,\dots,P_k})$ by left translations.
Let $f_1, \dots, f_n$ be a generating set for the free group F_n. If we set $\mu_m = \max_{1 \leq j \leq n} d_{A,P_1,\dots,P_k}(\varphi^m(f_j), id.)$ then by our assumptions, after possibly passing to a subsequence, $\mu_m \to \infty$.

Since restricted geodesics in the metric $d_{A,P_1,\dots,P_k}$ are quasi-geodesics in the word metric by lemma 4.8, and since the rescaling factors $\mu_m \to \infty$ we can extract from the sequence of actions of F_n on the metric spaces $(X, d_{A,P_1,\dots,P_k})$ a subsequence converging to a real tree Y by theorem 1.1 ([Pa1],2.3). The action of F_n on Y is stable and stabilizers of segments are either trivial or maximal cyclic by the proof of proposition 1.2 ([Ri-Se2],4.1-4.2). Furthermore, by construction the subgroup $H =< A, P_1, \dots, P_k >$ is a subgroup of a point stabilizer R in Y.

Since φ acts on the metric space $(X, d_{A,P_1,\dots,P_k})$ as a bi-Lipschitz equivariant map, and since $\mu_m \to \infty$, we can also construct a commutative diagram (1) from a converging subsequence of actions of F_n on the metric space $(X, d_{A,P_1,\dots,P_k})$ via powers of the automorphism φ. Since φ was assumed to have no periodic conjugacy classes, the commutative diagram (1) implies that Y contains no IET components and that the stabilizer of every segment in Y is trivial. Hence, the stabilizer of every point in Y is a periodic factor with respect to φ. Since H is invariant under φ so is the point stabilizer containing H, so $H < R$, R is a periodic factor properly contained in F_n, and R is invariant under the action of φ.

□

Propositions 4.10 and 4.13 conclude the proof of theorem 4.7 by a finite induction on the rank of the ambient group F_n.

□

Irreducible periodic factors form the basic level of our hierarchical decomposition. Irreducible bricks and Dehn extensions are the main objects that allow one to climb through the levels of the decomposition. Theorem 4.7 and lemma 4.6 show that an irreducible brick is a periodic factor and analyze the algebraic structure of an irreducible brick in terms of the periodic factors $P_1, \dots, P_k$, the irreducible extension of some of their subfactors A, and the additional conjugating elements $g_{k+1}, \dots, g_s$.

To climb up through the levels of the hierarchical decomposition we will need to understand finite collections of irreducible bricks and the subgroup generated by them. A basic property of irreducible bricks which turns out to be fundamental in understanding collections of them, is the classification of their periodic subfactors.

Proposition 4.14 *With the notation above let $H =< A, P_1, \dots, P_k, g_{k+1}, \dots, g_s >$ be an irreducible brick, suppose φ preserves both A, P_1 and the conjugacy classes of the periodic factors P_i, and let $D < H$ be a periodic subfactor in H. Then either:*

1) *D can be conjugated into one of the periodic factors P_i.*
2) *D can be conjugated in H so that $A < D$, there exist (non-conjugate) periodic subfactors of D: $L_{1,1}, \dots, L_{1,r_1}, \dots, L_{k,1}, \dots, L_{k,r_k}$, and elements $d_{r+1}, \dots, d_s \in D$ where $r = r_1 + \dots + r_k$ so that:*

(i) $M_j = A \cap L_{\alpha(j),\beta(j)}$ *for* $j = 1, \dots, r$, *and* $M_j = A \cap d_j L_{\alpha(j),\beta(j)} d_j^{-1}$ *for* $j = r+1, \dots, s$.

(ii) *the periodic factors* $L_{\alpha(j),\beta(j)}$ *can be conjugated into the periodic factor* $P_{\alpha(j)}$ *for* $j = 1, \dots, r$.

(iii) $L_{\alpha(j),\beta(j)}$ *intersects trivially every conjugate of* $L_{\alpha(j'),\beta(j')}$ *for* $1 \leq j < j' \leq r$.

(iv) $D =< A, L_{1,1}, \dots, L_{k,r_k}, d_{r+1}, \dots, d_s >$ *and* D *is an irreducible brick with respect to the irreducible extension* A, *the periodic factors* $L_{1,1}, \dots, L_{k,r_k}$ *and the conjugating elements* $d_{r+1}, \dots, d_s$.

Proof: To prove the proposition we will use the structure of the real trees Y_H and Y_H^1 constructed in the proof of lemma 4.6, and the commutative diagram (1) connecting between these two trees.

D is a subfactor of the irreducible brick H, so there exists a minimal subtree $Y_D < Y_H$ preserved by D. Since by the proof of lemma 4.6, all the point stabilizers in Y_H are conjugates of the periodic factors $P_1, \dots, P_k$, if Y_D is a single point, D can be conjugated into one of the periodic factors P_i, which is the first possibility stated in our proposition.

Suppose Y_D is not a single point. By possibly conjugating D we may assume that Y_D intersects Y_A in a non-degenerate segment. Since there are only finitely many orbits of germs issuing from branching points in Y_A by theorem 1.10, and since Y_A intersects Y_D in a non-degenerate segment, there exists an element $a_0 \in A$ which acts hyperbolically on Y_A so that $Y_D \cap a_0(Y_D)$ is non-degenerate. Since D is a periodic factor, proposition 3.1 implies that a_0 must also be an element in D, so $a_0 \in A \cap D$.
Now, since both A and D are periodic factors, $A \cap D$ is a peiodic factor as well. Since $a_0 \in A \cap D$ and a_0 acts hyperbolically on Y_A, $A \cap D$ can not be conjugated into one of the periodic factors $M_1, \dots, M_s$. Since, by our assumptions, A is an irreducible extension of the M_j's, and $A \cap D$ is a periodic subfactor in A, $A \cap D$ must be the entire irreducible extension A, so $A < D$.

Let $y_1, \dots, y_s \in Y_A$ be the (distinct) points stabilized by $M_1, \dots, M_s$ in correspondence. Since A is an irreducible extension of the M_j's the points $y_1, \dots, y_s$ represent all the distinct A-orbits of points with non-trivial stabilizer in Y_A, and the points $y_1, \dots, y_k$ represent all the H-orbits of points with non-trivial stabilizer in Y_A (and also in Y_H by the proof of lemma 4.6). Since $A < D < H$, we may assume w.l.o.g. that the points $y_1, \dots, y_r$ for some $k \leq r \leq s$ represent all the distinct D-orbits of points with non-trivial stabilizer in Y_A.

For each $1 \leq j \leq r$ let $1 \leq \alpha(j) \leq k$ indicates the index of a point from the set $y_1, \dots, y_k$ which is in the same H-orbit of y_j. Each H-orbit of points with non-trivial stabilizer is divided into finitely many D-orbits of such points. Suppose the H-orbit of $y_{\alpha(j)}$ is divided into $r_{\alpha(j)}$ D-orbits. We set $\beta(j)$, $1 \leq \beta(j) \leq r_j$ to indicate the D-orbit of y_j among all the points $y_1, \dots, y_r$ which belong to the same H-orbit as $y_{\alpha(j)}$.

Let $L_{\alpha(j),\beta(j)}$ be the stabilizer of y_j for $j = 1, \dots, r$. By the commutative diagram (1) $L_{\alpha(j),\beta(j)}$ is a periodic factor, by our definition of $\alpha(j)$, $L_{\alpha(j),\beta(j)}$ can be conjugated into $P_{\alpha(j)}$, and since the periodic factors $L_{\alpha(j),\beta(j)}$'s stabilize points from distinct D-orbits in Y_D for different indices j, $L_{\alpha(j),\beta(j)}$ intersects trivially every conjugate of $L_{\alpha(j'),\beta(j')}$ for $j \neq j'$.

Since , by our notation, for every index $j = r+1, \dots, s$, y_j is in the same D-orbit of $y_{j'}$ for some $1 \leq j' \leq r$ there exists an element d_j that maps $y_{j'}$ to y_j. Setting $\alpha(j') = \alpha(j)$ and $\beta(j') = \beta(j)$, D is by definition an irreducible brick with respect to the irreducible extension A, the periodic factors $L_{1,1}, \dots, L_{k,r_k}$ and the conjugating elements $d_{r+1}, \dots, d_s$, which is the second possibility in the proposition.

□

The structure of periodic subfactors of an irreducible brick gives some immediate corollaries on the correlations between different irreducible bricks, and the connections between irreducible bricks and Dehn extensions of the periodic factors $P_1, \dots, P_k$ it is composed from.

Lemma 4.15 *With the notation above let* $H = < A, P_1, \dots, P_k, g_{k+1}, \dots, g_s >$ *be an irreducible brick, suppose* φ *preserves both* A, P_1 *and the conjugacy classes of the periodic factors* P_i*. Then:*

(i) *If* P *is a periodic factor and* P *intersects trivially every conjugate of the periodic factors* $P_1, \dots, P_k$ *then* P *intersects trivially every conjugate of the irreducible brick* H*.*

(ii) *If* $GD(P_{i_0})$ *is the generalized Dehn closure of* P_{i_0} *in the ambient group* F_n *(see lemma 2.7 for this notion) then* $GD(P_{i_0}) \cap H = P_{i_0}$*. Furthermore, if* $GD(P_{i_0}) = P_{i_0} * < gd_1 > * \dots * < gd_u >$ *and* $L = < H, GD(P_{i_0}) >$*, then* L *is a periodic factor preserved by* φ*,* L *is an irreducible brick, and* $L = H * < gd_1 > * \dots * < gd_u >$*.*

(iii) *Let* $Q_1 = P_1$ *and* $Q_2, \dots, Q_{\hat{k}}$ *be periodic factors. Suppose* Q_i *intersects trivially every conjugate of* $Q_{i'}$ *for* $i \neq i'$*, and* Q_i *is either conjugate to one of the* P_j*'s or intersects trivially every conjugate of* P_j *for* $j = 1, \dots, k$*. Let* $\hat{A}$ *be an irreducible extension of subfactors of the periodic factors* $Q_1, \dots, Q_{\hat{k}}$ *and assume* $\hat{A}$ *is not conjugate to* A*. Let* $\hat{H}$ *be the irreducible brick containing* $\hat{A}$*,* $Q_1, \dots, Q_{\hat{k}}$ *and the corresponding conjugating elements* $\hat{g}_{\hat{k}+1}, \dots, \hat{g}_{\hat{s}}$*:*

$$\hat{H} = < \hat{A}, Q_1, \dots, Q_{\hat{k}}, g_{\hat{k}+1}, \dots, g_{\hat{s}} > \quad .$$

Then $\hat{H} \cap H = P_1 = Q_1$*.*

(iv) *Let* I *be an irreducible extension of the periodic factors* $N_1, \dots, N_c$ *and suppose at least one of the periodic factors* $N_1, \dots, N_c$ *can not be conjugated into any of the periodic factors* $P_1, \dots, P_k$*. Then the irreducible extension* I *can not be conjugated into the irreducible brick* H*.*

Proof: To prove (i) note that according to proposition 4.14, if D is a periodic subfactor of H then D intersects non-trivially a conjugate of at least one of the periodic factors $P_1, \dots, P_k$. Since P is assumed to intersect trivially every conjugate of $P_1, \dots, P_k$, if P' is any conjugate of P then $P' \cap H$ intersects trivially every conjugate of $P_1, \dots, P_k$. Since $P' \cap H$ is either trivial or a periodic subfactor of H, $P' \cap H$ must be trivial, and we get part (i) of the lemma.

Since A is an irreducible extension of periodic subfactors of $P_1, \dots, P_k$, lemma 4.3 implies that $M_{i_0} = A \cap P_{i_0} = A \cap GD(P_{i_0})$, and in particular A is not a subfactor of $GD(P_{i_0})$. Lemma 4.14 implies that a periodic subfactor of the irreducible brick H either contains a conjugate of A, or it can be conjugated into one of the periodic factors $P_1, \dots, P_k$. Hence, $GD(P_{i_0}) \cap H = P_{i_0}$.

Since the periodic factor P_{i_0} intersects trivially every conjugate of the periodic factor $P_{i'}$ for $i' \neq i_0$, lemma 2.8 implies that $GD(P_{i_0})$ intersects trivially every conjugate of $P_{i'}$ for every $i' \neq i_0$. If we set $W_{i'} = P_{i'}$ for every $i' = 1, \ldots, k$, $i' \neq i_0$ and $W_{i_0} = GD(P_{i_0})$, then A is an irreducible extension of periodic subfactors of $W_1, \ldots, W_k$, $A \cap W_i = A \cap P_i = M_i$ for $i = 1, \ldots, k$, and $A \cap g_j W_{i_j} g_j^{-1} = A \cap g_j P_{i_j} g_j^{-1} = M_j$ for $j = 1, \ldots, s$. Hence, the subgroup L generated by the irreducible extension A, the periodic factors $W_1, \ldots, W_k$, and the conjugating elements $g_{k+1}, \ldots, g_s$ is an irreducible brick that contains H. Furthermore, by the structure of irreducible bricks, derived in lemma 4.6, applied to both irreducible bricks H and L:

$$L \ = < H, GD(P_i) > = \ H * < gd_1 > * \ldots * < gd_u >$$

which gives us part (ii) of the lemma.

To prove part (iii) note that since the irreducible extensions $\hat{A}$ and A are not conjugate, lemma 4.4 implies that a conjugate of $\hat{A}$ either intersects A trivially, or it intersects it in a periodic factor that can be conjugated into one of the periodic factors $P_1, \ldots, P_k$, so in particular $\hat{A}$ contains no conjugate of A. Hence, lemma 4.14 implies that a conjugate of $\hat{A}$ intersects H either trivially or in a periodic factor that can be conjugated into one of the periodic factors $P_1, \ldots, P_k$, and in particular H contains no conjugate of $\hat{A}$. Similarly a conjugate of A intersects $\hat{H}$ either trivially or in a periodic factor that can be conjugated into one of the periodic factors $Q_1, \ldots, Q_{\hat{k}}$, and $\hat{H}$ contains no conjugate of A.

Let $D \ = \ \hat{H} \cap H$. $P_1 < D$ by construction, so if D properly contains P_1 then lemma 4.14, applies to both irreducible bricks H and $\hat{H}$, implies that D contains conjugates of both A and $\hat{A}$. Since D is a subfactor of both H and $\hat{H}$ it does not contain conjugates of A or $\hat{A}$, so necessarily $D = P_1$ and part (iii) follows.

To prove part (iv) suppose w.l.o.g. the irreducible extension $I < H$ and the periodic factor N_1 can not be conjugated into any of the periodic factors $P_1, \ldots, P_k$. Since by our assumptions N_1 is a periodic subfactor in the irreducible brick H, and N_1 can not be conjugated into any of the periodic factors $P_1, \ldots, P_k$, lemma 4.14 implies that N_1 can be conjugated in H so that $A < N_1$.

With the notation and construction used in lemma 4.14, let $y_1, \ldots, y_s$ be the points stabilized by $M_1, \ldots, M_s$ in Y_H, let $M_j < S_j$ be the stabilizer of y_j in N_1, and let $M_j < R_j$ be the stabilizer of y_j in I. Since the points stabilizers $\{R_j\}$ are periodic free factors properly contained in I, if for some $1 \leq j \leq k$ the stabilizer R_j of y_j in I properly contains the stabilizer S_j of y_j in N_1, R_j is a proper periodic factor contained in I that can not be conjugated into any of the periodic factors $N_1, \ldots, N_c$, a contradiction to I being an irreducible extension. Hence, for every $1 \leq j \leq k$, $R_j = S_j$.

If for some couple of indices $1 \leq j_1 < j_2 \leq s$, y_{j_1} and y_{j_2} belong to the same I-orbit in Y_H but to distinct N_1-orbits, R_{j_1} properly contains S_{j_1} which contradicts our last conclusion. Hence, I-orbits of the points $y_1, \ldots, y_s$ are identical with their N_1-orbits. Now, I and N_1 are both periodic subfactors in H, they both contain A, have identical points stabilizers and identical orbits of points with non-trivial stabilizers. Therefore, lemma 4.14 implies that $I = N_1$, a contradiction to I being an irreducible extension of the periodic factors $N_1, \ldots, N_c$, so I can not be conjugated into the irreducible brick H and we get part (iv).

□

Theorem 4.7 and lemma 4.6 study the algebraic structure of an irreducible brick. Part (iii) of lemma 4.15 studies the basic connection between "overlapping" non-conjugate irreducible bricks. To study the general structure of an automorphism we will need to look at finite collections of "overlapping" irreducible bricks which we call *irreducible chambers*.

Definition 4.16 *Let $P_1, \ldots, P_k$ be periodic factors with respect to an automorphism φ and suppose P_i intersects trivially every conjugate of $P_{i'}$ for $i \neq i'$. Let $A^1, \ldots, A^r$ be pairwise non-conjugate irreducible extensions of subfactors of $P_1, \ldots, P_k$, $A^\ell = B^\ell * M_1^\ell * \ldots * M_{s_\ell}^\ell$ for $\ell = 1, \ldots, r$, where the periodic factor M_j^ℓ can be conjugated into the periodic factor $P_{i(\ell,j)}$ for $j = 1, \ldots, s_\ell$.*

Let $\Delta = \Delta(A^1, \ldots, A^r, P_1, \ldots, P_k)$ be a (finite) bi-partite graph with one type of vertices corresponding to the periodic factors $P_1, \ldots, P_k$, and second type of vertices corresponding to the irreducible extensions of their subfactors $A^1, \ldots, A^r$. To each ordered couple (ℓ, j), where $\ell = 1, \ldots, r$ and $j = 1, \ldots, s_\ell$, we associate an edge $e_{(\ell,j)}$ in Δ connecting the irreducible extension A^ℓ with the periodic factor $P_{i(\ell,j)}$ (note that in general Δ is not a simple graph). For the collection of periodic factors P_i and irreducible extensions A^ℓ to be an irreducible chamber we require that the graph $\Delta = \Delta(A^1, \ldots, A^r, P_1, \ldots, P_k)$ is connected.

Suppose the graph Δ is connected and let T be a maximal tree in Δ. By conjugating appropriately the periodic factors P_i and the irreducible extensions $A^1, \ldots, A^r$ we may assume that if $e_{(\ell,j)} \in T$ then $M_j^\ell = A^\ell \cap P_{i(\ell,j)}$ (i.e., the last intersection is not only conjugate to M_j^ℓ but it is M_j^ℓ itself). For each couple (ℓ, j) for which $e_{(\ell,j)} \notin T$ we associate a conjugating element $v_{(\ell,j)}$ for which $M_j^\ell = v_{(\ell,j)} P_{i(\ell,j)} v_{(\ell,j)}^{-1} \cap A^\ell$ (the coset $v_{(\ell,j)} P_{i(\ell,j)}$ is clearly uniquely defined given the periodic factors $P_{i(\ell,j)}$ and M_j^ℓ). We call the subgroup generated by the periodic factors $P_1, \ldots, P_k$, the irreducible extensions $A^1, \ldots, A^r$ (conjugating in accordance with the maximal tree T), and the conjugating elements $\{v_{(\ell,j)}\}$ corresponding to all edges $e_{(\ell,j)} \notin T$ an irreducible chamber, and denote it $CH(A^1, \ldots, A^r, P_1, \ldots, P_k, \{v_{(\ell,j)}\})$.

A standard argument, which we leave for the interested reader, shows that the conjugacy class of an irreducible chamber does not depend on the particular tree T chosen for its definition. The structure of irreducible bricks given by lemma 4.6 and theorem 4.7, and their intersection properties given by lemma 4.15 allow us to obtain the structure of irreducible chambers.

Theorem 4.17 *With the notation and assumptions of definition 4.16, an irreducible chamber $CH(P_1, \ldots, P_k, A^1, \ldots, A^r, \{v_{(\ell,j)}\})$ is a periodic factor with re-*

spect to φ and:

$$CH(A^1,\dots,A^r,P_1,\dots,P_k,\{v_{(\ell,j)}\}) \;=\; B^1{*}\dots{*}B^r{*}P_1{*}\dots{*}P_k{*}\Big({*}_{e_{(\ell,j)}\notin T} < v_{(\ell,j)} >\Big)$$

Proof: To get the algebraic structure of an irreducible chamber we represent it as a (finite) iterated sequence of irreducible bricks. We will use the notation of definition 4.16.
Since the graph $\Delta(A^1,\dots,A^r,P_1,\dots,P_k)$ is connected, it is possible to rearrange the irreducible extensions $A^1,\dots,A^r$ and the periodic factors $P_1,\dots,P_k$, so that there will exist a non-decreasing sequence of integers $1 \leq k_1 \leq \dots \leq k_r = k$, for which for every fixed $1 \leq \ell \leq r$ the partial graph $\Delta^\ell \subset \Delta$ containing all the edges from $A^1,\dots,A^\ell$ to $P_1,\dots,P_{k_\ell}$ is connected, and Δ^ℓ contains all the edges issuing from the vertices $A^1,\dots,A^\ell$ in Δ. Furthermore, we can choose the maximal tree $T_\Delta \subset \Delta$ so that $T_{\Delta^\ell} = T_\Delta \cap \Delta^\ell$ is a maximal tree in Δ^ℓ.
By construction, for a fixed $1 \leq \ell \leq k$, the subgroup CH^ℓ generated by the irreducible extensions $A^1,\dots,A^\ell$, the periodic factors $P_1,\dots,P_{k_\ell}$ and the conjugating elements $\{v_{(m,j)}\}$ corresponding to all edges $e_{(m,j)} \in \Delta^\ell$, $e_{(m,j)} \notin T_{\Delta^\ell}$ is an irreducible chamber.

CH^1 is an irreducible brick containing the irreducible extension A^1, the periodic factors $P_1,\dots,P_{k_1}$ and the conjugating elements $\{v_{(1,j)}\}$ corresponding to edges $e_{(1,j)} \notin T$. Property (i) in lemma 4.15 implies that a conjugate of each of the periodic factors $P_{k_1+1},\dots,P_k$ intersects CH^1 trivially. Combining that with property (iii) in lemma 4.15, CH^2 is an irreducible brick containing the irreducible extension A^2, the periodic factors $CH^1, P_{k_1+1},\dots,P_{k_2}$, and the conjugating elements $\{v_{(2,j)}\}$ corresponding to edges $e_{(2,j)} \notin T$. Continuing with a finite induction, properties (i) and (iii) of lemma 4.15 imply that CH^ℓ is an irreducible brick containing the irreducible extension A^ℓ, the periodic factors $CH^{\ell-1}, P_{k_{\ell-1}+1},\dots,P_{k_\ell}$, and the conjugating elements $\{v_{(\ell,j)}\}$ corresponding to edges $e_{(\ell,j)} \notin T$.

Since we have managed to represent the irreducible chamber $CH(A^1,\dots,A^r,P_1,\dots,P_k,\{v_{(\ell,j)}\})$ as an iterated sequence of irreducible bricks, it is a periodic factor with respect to φ by theorem 4.7. Its algebraic structure, as stated in the theorem, follows by iterative applications of lemma 4.6.

□

Since irreducible chambers can be viewed as an iterative construction of irreducible bricks, the basic properties satisfied by irreducible bricks (lemma 4.15) naturally hold for irreducible chambers.

Lemma 4.18 *With the notation of definition 4.11 and theorem 4.17, let $CH =< A^1,\dots,A^r,P_1,\dots,P_k,\{v_{(\ell,j)}\} >$ be an irreducible chamber, suppose (w.l.o.g.) that φ preserves A^1 and P_1, and the conjugacy classes of the periodic factors P_i and the irreducible extensions A^ℓ. Then:*

- (i) *If P is a periodic factor and P intersects trivially every conjugate of the periodic factors $P_1,\dots,P_k$ then P intersects trivially every conjugate of the irreducible chamber CH.*
- (ii) *Let $GD(P_i)$ be the generalized Dehn closure of P_i in the ambient group F_n (see lemma 2.7 for this notion). Then $GD(P_i) \cap CH = P_i$. Furthermore, if $GD(P_i) = P_i{*} < gd^i_1 > {*}\dots{*} < gd^i_{u_i} >$ and $L =< CH, GD(P_1),\dots,GD(P_k) >$,*

then L is a periodic factor preserved by φ, L is an irreducible chamber, and:

$$L = CH * < gd_1^1 > * \ldots * < gd_{u_1}^1 > * < gd_1^2 > * \ldots * < gd_{u_k}^k > \quad .$$

(iii) *Let $Q_1 = P_1$ and $Q_2, \ldots, Q_{\hat{k}}$ be periodic factors. Suppose Q_i intersects trivially every conjugate of $Q_{i'}$ for $i \neq i'$, and Q_i is either conjugate to one of the P_j's or intersects trivially every conjugate of P_j for $j = 1, \ldots, k$. Let $\hat{A}^1, \ldots, \hat{A}^{\hat{r}}$ be pairwise non-conjugate irreducible extensions of subfactors of the periodic factors $Q_1, \ldots, Q_{\hat{k}}$ and assume $\hat{A}_{\hat{\ell}}$ is not conjugate to any of the irreducible extensions $A^1, \ldots, A^r$.*
With the notation of definition 4.16 suppose the graph $\Delta(\hat{A}^1, \ldots, \hat{A}^{\hat{r}}, Q_1, \ldots, Q_{\hat{k}})$ is connected and let $\hat{T}$ be a maximal tree in $\hat{\Delta}$. Let $\hat{CH}$ be the irreducible brick containing the irreducible extensions $\hat{A}^1, \ldots, \hat{A}^{\hat{r}}$, the periodic factors $Q_1, \ldots, Q_{\hat{k}}$ and the conjugating elements $\{\hat{v}_{\hat{\ell},\hat{j})}\}$ corresponding to edges $\hat{e}_{(\hat{\ell},\hat{j})}$ in $\hat{\Delta}$ which do not belong to $\hat{T}$:

$$\hat{CH} =< \hat{A}^1, \ldots, \hat{A}^{\hat{r}}, Q_1, \ldots, Q_{\hat{k}}, \{\hat{v}_{(\hat{\ell},\hat{j})}\} > \quad .$$

Then $\hat{CH} \cap CH = P_1 = Q_1$.

(iv) *Let I be an irreducible extension of the periodic factors $N_1, \ldots, N_c$ and suppose at least one of the periodic factors $N_1, \ldots, N_c$ can not be conjugated into any of the periodic factors $P_1, \ldots, P_k$. Then the irreducible extension I can not be conjugated into the irreducible chamber CH.*

Proof: With the notation of the lemma and those of the proof of theorem 4.17, part (i) of lemma 4.15 implies that P intersects trivially the irreducible brick CH^1, and continuing by finite induction P intersects trivially the irreducible bricks $CH^2, \ldots, CH^r = CH$ and we get part (i).

By part (ii) of lemma 4.15 $L^1 =< CH^1, GD(P_1), \ldots, GD(P_{k_1}) >$ is a Dehn extension of the irreducible brick CH^1 and:

$$L^1 = CH^1 * < gd_1^1 > * \ldots * < gd_{u_1}^1 > * < gd_1^2 > * \ldots * < gd_{u_{k_1}}^{k_1} > \quad .$$

Since the irreducible chamber CH can be represented as an iterated sequence of irreducible bricks $CH^1, \ldots, CH^r = CH$, part (ii) of the lemma follows by applying part (ii) of lemma 4.15 repeatedly to that iterative sequence.

To prove part (iii) of the lemma we may first reorder the periodic factors Q_i so that each of the periodic factors $Q_1, \ldots, Q_b$ are conjugates of one of the periodic factors $P_1, \ldots, P_k$, and each of the periodic factors $Q_{b+1}, \ldots, Q_{\hat{k}}$ intersects trivially every conjugate of one of the periodic factors $P_1, \ldots, P_k$. Also, by our assumptions each of the irreducible extensions $\hat{A}^1, \ldots, \hat{A}^{\hat{r}}$ is not conjugate to any of the irreducible extensions $A^1, \ldots, A^r$. Hence, with the notation of definition 4.16, the graph:

$$\tilde{\Delta} = \Delta(A^1, \ldots, A^r, \hat{A}^1, \ldots, \hat{A}^{\hat{r}}, P_1, \ldots, P_k, Q_{b+1}, \ldots, Q_{\hat{k}})$$

is connected, so the subgroup UCH generated by the irreducible extensions $A^1, \ldots, A^r, \hat{A}^1, \ldots, \hat{A}^{\hat{r}}$, the periodic factors $P_1, \ldots, P_k, Q_{b+1}, \ldots, Q_{\hat{k}}$ and the conjugating elements $\{v_{(\tilde{\ell},\tilde{j})}\}$ corresponding to edges lying outside a maximal tree in $\tilde{\Delta}$

is an irreducible chamber. Now, part (iii) of the lemma follows from the structure of the irreducible chambers CH, $\hat{CH}$ and UCH given by theorem 4.17 (the details are left for the interested reader).

To prove part (iv) of the lemma note that by part (iv) of lemma 4.15 the irreducible extension I can not be conjugated into the irreducible brick CH^1. If at least one of the periodic factors $N_1, \ldots, N_c$ can not be conjugated into any of the periodic factors $CH^1, P_{k_1+1}, \ldots, P_{k_2}$, part (iv) of lemma 4.15 implies that I can not be conjugated into the irreducible brick CH^2. If each of the periodic factors $N_1, \ldots, N_c$ can be conjugated into one of the periodic factors $CH^1, P_{k_1+1}, \ldots, P_{k_2}$, I is an irreducible extension of subfactors of the periodic factors $CH^1, P_{k_1+1}, \ldots, P_{k_2}$ which is not conjugate to to the irreducible extension A^2. Hence, the irreducible extensions I and A^2, together with the periodic factors $CH^1, P_{k_1+1}, \ldots, P_{k_2}$ and appropriate conjugating elements, form an irreducible chamber, and the algebraic structure of irreducible chambers proven in theorem 4.17 shows that I can not be conjugated into the irreducible brick CH^2. Repeating these two arguments inductively we may conclude that I can not be conjugated into any of the irreducible bricks $CH^3, \ldots, CH^r = CH$ and part (iv) follows.

□

Theorem 4.17 shows that an irreducible chamber is a periodic factor and determine its algebraic structure in terms of the periodic factors $P_1, \ldots, P_k$, the irreducible extensions of their subfactors $A^1, \ldots, A^r$ and the additional conjugating elements. Since the rank of a periodic factor is bounded by the rank of the ambient group F_n, theorem 3.11 together with theorem 4.17 imply that given periodic factors $P_1, \ldots, P_k$ there can be only finitely many pairwise non-conjugate irreducible extensions of their subfactors.

Definition 4.19 *Let $P_1, \ldots, P_k$ be periodic factors with respect to an automorphism φ and suppose P_i intersects trivially every conjugate of $P_{i'}$ for $i \neq i'$. If the (finite) set of all pairwise non-conjugate irreducible extensions of subfactors of $P_1, \ldots, P_k$ defines an irreducible chamber (i.e., the corresponding graph defined in definition 4.16 is connected) we call this irreducible chamber the irreducible closure of the periodic factors $P_1, \ldots, P_k$ and denote it $IC(P_1, \ldots, P_k)$.*

Note that since irreducible closures are in particular irreducible chambers, all the properties that hold for irreducible chambers, and in particular those stated in lemma 4.18, hold for irreducible closures.

5. The Hierarchical Decomposition.

In section 2 we have defined periodic and irreducible free factors, and their Dehn and irreducible extensions. In section 3 we have analyzed the algebraic connections between distinct periodic factors, and in section 4 we have analyzed irreducible extensions of periodic subfactors and shown that irreducible bricks, chambers and closures are periodic free factors. In all these sections we have assumed that there are no periodic conjugacy classes with respect to the automorphism in question.
In this section we combine the notions and structural results introduced in the previous sections to construct our (canonical) hierarchical decomposition for an automorphism with no periodic conjugacy classes, and derive some of its basic properties. Since a periodic factor contains no periodic conjugacy classes, the hierarchical decomposition we construct in this section gives, in particular, a canonical hierarchical structure to periodic factors with respect to general automorphisms. The

general analysis of automorphisms that admit periodic conjugacy classes and their hierarchical decomposition, which combines the basic canonical graph of groups derived in ([Se1],4.1) with structural results obtained in this paper, is conducted in a continuation paper.

Throughout this section we will assume that the automorphism φ in question admits no periodic conjugacy classes. To define our hierarchical decomposition we need to introduce some (canonical) periodic factors with respect to φ which we call *huts* and *blocks*. To each hut and each block there is an associated *level*, the number of levels is bounded by the rank of the ambient group F_n, and for each level there are only finitely many huts and blocks. The level 1 huts are the irreducible factors with respect to φ, and the (unique) maximal level block is the ambient group F_n.

Huts and blocks are defined iteratively. We will first define the huts and blocks of level 1 and then define the huts and blocks at level $\ell+1$ using the (finite) set of blocks at level ℓ. The complete list of huts and blocks from all levels basically concludes the construction of the hierarchical decomposition, and the rest of the section is devoted to some of the basic properties of the decomposition obtained.

Lemma 5.1 *Let φ be an automorphism with no periodic conjugacy classes of a free group F_n. There exist finitely many conjugacy classes of irreducible (periodic) factors $R_1,\ldots,R_s$ with respect to φ, φ permutes the conjugacy classes of the irreducible factors, and after possibly replacing the R_i's with appropriate conjugates, $R_1 * \ldots * R_s$ is a free factor in F_n.*

Proof: Since the intersection of periodic factors is either trivial or a periodic factor, if R and R' are non-conjugate irreducible factors with respect to φ then R intersects trivially every conjugate of R'. Hence, theorem 3.11 implies that if $R_1,\ldots,R_i$ is a set of pairwise non-conjugate irreducible factors with respect to φ, then after possibly replacing them by appropriate conjugates, $R_1 * \ldots * R_i$ is a free factor in F_n, and in particular, there are only finitely many conjugacy classes of irreducible factors with respect to φ. Since if R is an irreducible factor so is $\varphi(R)$, φ permutes the conjugacy classes of its irreducible factors.

□

The *huts of level* ℓ, $Hu^{\ell}_1,\ldots,Hu^{\ell}_{q_\ell}$, is a canonically defined collection of periodic factors with respect to φ for which Hu^{ℓ}_i intersects trivially every conjugate of $Hu^{\ell}_{i'}$ for every $i' \neq i$. We define the set of *huts of level* 1, $Hu^1_1,\ldots,Hu^1_{q_1}$ to be the complete set of pairwise non-conjugate irreducible factors with respect to φ. W.l.o.g. we assume that $Hu^1_1 * \ldots * Hu^1_{q_1}$ is a free factor in F_n. In general, once we set the huts of certain level, to obtain the blocks at that level we unify disjoint subsets of huts.

Definition 5.2 *Let $P_1,\ldots,P_k$ be periodic factors with respect to φ so that P_i intersects trivially every conjugate of $P_{i'}$ for every $i' \neq i$. From the set $P_1,\ldots,P_k$ we obtain a new set of periodic factors $Q_1,\ldots,Q_s$ by a canonical operation which we call $unifying$ periodic $factors$.*

If for some conjugate $\hat{P}_{i'}$ of $P_{i'}$, $P_i * \hat{P}_{i'}$ is a periodic factor with respect to φ, we replace P_i with $P_i * \hat{P}_{i'}$ in our list and erase $P_{i'}$ from it. We repeat unifying periodic factors as long as we can, and obtain the list $Q_1,\ldots,Q_s$.
Since, by theorem 3.11, we may assume that $P_1 * \ldots * P_k$ is a free factors in F_n, Q_j intersects trivially every conjugate of $Q_{j'}$ for $j \neq j'$, and the conjugacy classes

of the obtained periodic factors $Q_1,\dots,Q_s$ depend only on the periodic factors $P_1,\dots,P_k$ we have started with and not on the order of the unifications. Also, for every $j' \neq j$ and every conjugate $\hat{Q}_{j'}$ of $Q_{j'}$, the subgroup $Q_j * \hat{Q}_{j'}$ is not a periodic factor with respect to φ. Furthermore, by basic properties of automorphisms of a free product ([F-R1],[F-R2]), or alternatively by the commutative diagram (1), there is no periodic factor $S < F_n$ so that S is not conjugate to one of the Q_j's and $S = \hat{Q}_{j_1} * \dots * \hat{Q}_{j_m}$ where $\hat{Q}_{j_u}$ is a conjugate of Q_{j_u}. By theorem 3.11, after properly conjugating the periodic factors $\{P_i\}$ and $\{Q_j\}$ we may assume that $P_1 * \dots * P_k = Q_1 * \dots * Q_s$ is a free factor in F_n.

The irreducible factors with respect to φ form the basic level of our hierarchical decomposition, the huts of level 1. To climb to the second level, and in general to climb from level ℓ to level $\ell+1$ we need to introduce *blocks*. The blocks of a given level are periodic factors obtained from the huts of the same level by unifying periodic factors.

Definition 5.3 *Let $Hu_1^\ell,\dots,Hu_{q_\ell}^\ell$ be the set of huts of level ℓ. We assume that Hu_i^ℓ intersects trivially every conjugate of $Hu_{i'}^\ell$ for $i' \neq i$. The blocks of level ℓ, $BL_1^\ell,\dots,BL_{p_\ell}^\ell$, is the set of periodic factors obtained from the set of huts of level ℓ by unifying periodic free factors (definition 5.2 above).*

Clearly, $p_\ell \leq q_\ell$. By the properties of unified periodic factors listed in proposition 5.2, after properly conjugating the huts and blocks of certain level, we may assume that $Hu_1^\ell * \dots * Hu_{q_\ell}^\ell = BL_1^\ell * \dots * BL_{p_\ell}^\ell$ is a free factor in F_n. Also, there is no periodic factor $S < F_n$ so that S is not conjugate to one of the blocks of level ℓ and $S = \hat{BL}_{j_1} * \dots * \hat{BL}_{j_m}$ where $\hat{BL}_{j_u}$ is a conjugate of BL_{j_u}.

Having defined the huts of level 1, and obtaining the set of blocks of level ℓ from the set of huts at that level, we are ready to define (iteratively) the huts of higher levels.
Let $BL_1^\ell,\dots,BL_{p_\ell}^\ell$ be the set of blocks of level ℓ. By theorems 4.17 and 3.11, there can be only finitely many pairwise non-conjugate irreducible extensions of subfactors of blocks of level ℓ, so let $A_1^\ell,\dots,A_{r_\ell}^\ell$ be the entire set of pairwise non-conjugate irreducible extensions of periodic subfactors of blocks of level ℓ. Let Θ^ℓ be a bi-partite graph with one type of vertices corresponding to the blocks of level ℓ $\{BL_j^\ell\}$, and second type of vertices corresponding to the entire set of pairwise non-conjugate irreducible extensions of subfactors of the blocks of level ℓ $\{A_m^\ell\}$. The vertex corresponding to A_m^ℓ is connected by an edge to the vertex corresponding to BL_j^ℓ if and only if the irreducible extension A_j^ℓ intersects non-trivially a conjugate of the block BL_j^ℓ. Let $\Theta_1^\ell,\dots,\Theta_{s_\ell}^\ell$ be the connected components of Θ^ℓ. Clearly, $s_\ell \leq p_\ell$. W.l.o.g. we may assume that the vertices in Θ^ℓ corresponding to $BL_1^\ell,\dots,BL_{k_1}^\ell$ are in Θ_1^ℓ, the vertices in Θ^ℓ corresponding to $BL_{k_1+1}^\ell,\dots,BL_{k_2}^\ell$ are in Θ_2^ℓ etc. .

Definition 5.4 *With the notation above let $GD(BL_j^\ell)$ be the generalized Dehn closure of the block BL_j^ℓ for $j=1,\dots,q_\ell$, and let $IC_i^\ell = IC(BL_{k_{i-1}+1}^\ell,\dots,BL_{k_i}^\ell)$ be the irreducible closure of the blocks $BL_{k_{i-1}+1}^\ell,\dots,BL_{k_i}^\ell$ for $i=1,\dots,s_\ell$. We set $q_{\ell+1}=s_\ell$ and for $i=1,\dots,q_{\ell+1}$ we define the i-th hut of level $\ell+1$, $Hu_i^{\ell+1}$, to be the subgroup generated by the irreducible closure of the blocks $BL_{k_{i-1}+1}^\ell,\dots,BL_{k_i}^\ell$ and their generalized Dehn closures:*

$$Hu_i^{\ell+1} = < IC_i^{\ell+1}, GD(BL_{k_{i-1}+1}^\ell),\dots,GD(BL_{k_i}^\ell) > \ .$$

Note that a block of level ℓ, BL_j^{ℓ}, can be identical with a hut of level $\ell+1$, $Hu_i^{\ell+1}$, if BL_j^{ℓ} admits no Dehn extensions and there is no irreducible extension of periodic subfactors of blocks of level ℓ, A_m^{ℓ}, that intersects non-trivially a conjugate of BL_j^{ℓ}.

To show that the iterative setting of huts and blocks in higher levels given in definitions 5.3 and 5.4 is well defined we need to show that huts of all levels are periodic factors and that Hu_i^{ℓ} intersects trivially every conjugate of $Hu_{i'}^{\ell}$ for every $i' \neq i$.

Proposition 5.5 *With the notation of definitions 5.3 and 5.4 above:*

(i) $q_1 \geq p_1 \geq q_2 \geq p_2 \geq \dots$

(ii) *a hut is a periodic factor with respect to φ.*

(iii) *Hu_i^{ℓ} intersects trivially every conjugate of $Hu_{i'}^{\ell}$ for every ℓ and $i' \neq i$.*

(iv) *φ acts as a permutation on the sets of conjugacy classes of huts and blocks of a given level ℓ.*

(v) *The ambient group F_n is the unique block of the highest level ℓ_0, where $\ell_0 \leq \frac{2}{3}n$.*

Proof: $q_\ell \geq p_\ell$ since blocks of level ℓ are obtained from huts of the same level by unifying periodic factors. $p_\ell \geq q_{\ell+1}$ since $q_{\ell+1} = s_\ell$ is the number of connected components in the graph Θ^{ℓ}, and each connected component Θ_i^{ℓ} contains at least one vertex corresponding to a block of level ℓ. This proves part (i) of the proposition. We prove parts (ii) and (iii) by induction on the level. We assume all huts of level ℓ satisfy properties (ii) and (iii) and prove huts of level $\ell+1$ satisfy them as well. Both properties hold for irreducible factors which are the huts of level 1.

A hut $Hu_i^{\ell+1}$ is generated by the irreducible closure IC_i^{ℓ} of the blocks $BL_{k_{i-1}+1}^{\ell}, \dots, BL_{k_i}^{\ell}$ and their generalized Dehn closures $GD(BL_{k_{i-1}+1}^{\ell}), \dots, GD(BL_{k_i}^{\ell})$. By lemma 2.7 the generalized Dehn closure of a periodic factor is a periodic factor, and by lemma 2.8 and our inductive hypothesis $GD(BL_i^{\ell})$ intersects trivially $GD(BL_{i'}^{\ell})$ for every $i' \neq i$. Hence, the hut $Hu_i^{\ell+1}$ is an irreducible chamber generated by the generalized Dehn closures $GD(BL_{k_{i-1}+1}^{\ell}), \dots, GD(BL_{k_i}^{\ell})$, the irreducible extensions corresponding to vertices in the connected component Θ_i^{ℓ} of the graph Θ^{ℓ} and appropriate conjugating elements. Since irreducible chambers are periodic factors by theorem 4.17, the hut $Hu_i^{\ell+1}$ is a periodic factor.

Let $i' \neq i$, $1 \leq i' \leq q_{\ell+1}$. Since the blocks $BL^{\ell}_{k_{i'-1}+1}, \dots, BL^{\ell}_{k_{i'}}$ intersect trivially every conjugate of each of the blocks $BL^{\ell}_{k_{i-1}+1}, \dots, BL^{\ell}_{k_i}$ by our inductive hypothesis, so do their generalized Dehn closures $GD(BL^{\ell}_{k_{i'-1}+1}), \dots, GD(BL^{\ell}_{k_{i'}})$ by lemma 2.8. Since in addition the hut $Hu^{\ell+1}_i$ is an irreducible chamber, part (i) of lemma 4.18 implies that the generalized Dehn closures $GD(BL^{\ell}_{k_{i'-1}+1}), \dots, GD(BL^{\ell}_{k_{i'}})$ intersect trivially every conjugate of the hut $Hu^{\ell+1}_i$. Now, since the hut $Hu^{\ell+1}_{i'}$ is an irreducible chamber as well, part (i) of lemma 4.18 implies that the hut $Hu^{\ell+1}_i$ intersects trivially every conjugate of the hut $Hu^{\ell+1}_{i'}$, and parts (ii) and (iii) of the proposition follow by induction.

Since φ permutes the conjugacy classes of irreducible factors and φ maps a periodic factor to a periodic factor, if A is an irreducible extension of subfactors of some periodic factors $P_1, \dots, P_k$ then $\varphi(A)$ is an irreducible extension of subfactors of $\varphi(P_1), \dots, \varphi(P_k)$, and if $GD(P)$ is the generalized Dehn closure of a periodic factor P then $\varphi(GD(P))$ is the generalized Dehn closure of $\varphi(P)$, φ maps a hut of level ℓ to a hut of level ℓ, and a block of level ℓ to a block of level ℓ which gives us part (iv) of the proposition.

We prove part (v) by induction on the rank of the ambient group F_n. Note that there is no cyclic periodic factors with respect to φ, since φ is an automorphism with no periodic conjugacy classes. A periodic factor of rank 2 is always irreducible. Also, note that if the ambient group F_n is an irreducible factor then the ambient group is a block of level 1, and the claim of part (v) is obvious. Hence, we may assume that F_n is not an irreducible factor with respect to φ.

By theorem 3.11 and properties (ii) and (iii) of this proposition, we may properly conjugate the blocks of level ℓ and assume that $BL^{\ell}_1 * \dots * BL^{\ell}_{p_\ell}$ is a free factor in F_n. Also, by our construction of huts and blocks, as long as at least one of the blocks of level ℓ admits a Dehn extension or there exists an irreducible extension with respect to subfactors of blocks of level ℓ:

$$rk(BL^{\ell+1}_1 * \dots * BL^{\ell+1}_{p_{\ell+1}}) > rk(BL^{\ell}_1 * \dots * BL^{\ell}_{p_\ell}) .$$

Hence, if at no level the ambient group F_n is a block, there must exist some level ℓ_t so that all blocks of level ℓ_t admit no Dehn extensions, and there is no irreducible extension of subfactors of blocks of level ℓ_t.

If every periodic factor P properly contained in F_n can be conjugated into one of the blocks of level ℓ_t, then, by definition, F_n is an irreducible extension of the blocks of level ℓ_t, which contradicts our last conclusion. Hence, there must exist a periodic factor P properly contained in F_n, so that P can not be conjugated into any of the blocks of level ℓ_t.
P can not be an irreducible factor, since an irreducible factor is a hut of level 1, and a hut of level ℓ is a subfactor of one of the blocks at any higher level by construction, so an irreducible factor can be conjugated into one of the blocks of level ℓ_t. Since the rank of P is strictly smaller than that of F_n, and a power of φ composed with an inner automorphism preserves the periodic factor P, we may apply our inductive hypothesis and obtain a set of huts and blocks in P, $Hu(P)^{\ell}_i$ and $BL(P)^{\ell}_j$, so that for some $\ell_0(P)$, P is the unique block of level $\ell_0(P)$ in P, i.e., $BL(P)^{\ell_0(P)}_1 = P$.

Huts of level 1 in P are irreducible factors so they are also huts of level 1 in F_n. Since blocks are obtained from huts by unifying periodic factors, all blocks of level 1 in P are subfactors of blocks of level 1 in F_n. Hence, by the construction of huts of level 2 from blocks of level 1, all huts of level 2 in P are subfactors of huts of level 2 in F_n, and all blocks of level 2 in P are subfactors of blocks of level 2 in F_n, and by straightforward induction all huts of level ℓ in P are subfactors of huts of level ℓ in F_n and all blocks of level ℓ in P are subfactors of blocks of level ℓ in F_n. Now, P is a block of level $\ell_0(P)$ in P by our hypothesis, so P is a subfactor of a block of level $\ell_0(P)$ in F_n. Since, by our assumptions, every block in F_n can be conjugated into a block of level ℓ_t in F_n, P can be conjugated into a block of level ℓ_t in F_n, a contradiction to the choice of P, so the ambient group F_n must be a block of some level ℓ_0 which proves (v).

□

The *hierarchical decomposition of F_n associated with φ* is the entire collection of huts and blocks from level 1 (the irreducible factors) up to level ℓ_0 (the ambient group F_n) together with the way they are obtained as Dehn closures, irreducible chambers, and unified periodic factors. We call ℓ_0 the *level of the automorphism* φ. Since a block of level ℓ is contained in some block of level ℓ' for every $\ell' > \ell$, and since the ambient group F_n is a block of level ℓ_0 by the above proposition, for every periodic factor P there exists a level $\ell(P)$ for which P can be conjugated into some block of level $\ell(P)$ and P can not be conjugated into any of the blocks of level strictly smaller than $\ell(P)$. We call $\ell(P)$ the *level of the periodic factor P*.

By its construction, the hierarchical decomposition is canonical, and the hierarchical decomposition associated with φ is identical with the one associated with all powers of φ. The hierarchical decomposition of an automorphism with no periodic conjugacy classes remains invariant if we compose the given automorphism with an inner one, hence, it is in fact associated with elements of the outer automorphism group of F_n. From our point of view, which is stressed and clarified in [Se1], the hierarchical decomposition associated with an automorphism of a free group is the analogue of the Nielsen-Thurston classification of automorphsims of surfaces (which is also a special case of the hierarchical decomposition for general auto. of free groups as shown in our continuation paper). To study general combinatorial properties of automorphisms of free groups, our hierarchical decomposition should be combined with the train-tracks and invariant laminations of M. Bestvina and M. Handel [Be-Ha]. We continue this section by proving some basic properties of the decomposition which shed some light on the information it carries.

Proposition 5.6 *With the notation above, for every irreducible extension A in F_n there exists a level ℓ, so that A is conjugate to an irreducible extension A^{ℓ}_m which appears in one of the irreducible chambers defining a hut of level $\ell + 1$.*

Proof: Let $\ell(A)$ be the level of the irreducible extension A, i.e., w.l.o.g. A is a subfactor of $BL_1^{\ell(A)}$ and A can not be conjugated into any of the blocks of lower level. Since blocks are obtained from huts by unifying periodic factors, and since A is an irreducible extension, lemma 4.5 implies that A can be conjugated into a hut of level $\ell(A)$. W.l.o.g. A is a subfactor of $Hu_1^{\ell(A)}$.
Let A be an irreducible extension of the periodic factors $N_1, \ldots, N_c$, i.e., $A = B * N_1 * \ldots * N_c$ for some free factor B. Since the hut $Hu_1^{\ell(A)}$ is an irreducible chamber, and A is a subfactor of $Hu_1^{\ell(A)}$, part (iv) of lemma 4.18 implies that all the periodic factors $N_1, \ldots, N_c$ can be conjugated into the generalized Dehn closures of blocks of level $\ell(A) - 1$. Since A itself can not be conjugated into any of the blocks of level $\ell(A) - 1$ and A is an irreducible extension, lemma 4.3 implies

that A can not be conjugated into any of the generalized closures of the blocks of level $\ell(A)-1$. Now, since A can be conjugated into the hut $Hu_1^{\ell(A)}$ which is an irreducible chamber, A can not be conjugated into any of the generalized Dehn closures of blocks of level $\ell(A)-1$, and A is an irreducible extension of subfactors of these generalized Dehn closures, the structure theorem for irreducible chambers (theorem 4.17) implies that A is conjugate to one of the irreducible extensions defining the irreducible chamber $Hu_1^{\ell(A)}$, i.e., A is conjugate to some irreducible extension $A_m^{\ell(A)-1}$ appears in the hierarchical decomposition and the proposition follows.

□

The graph of groups Λ_φ constructed in ([Se1],4.1) gives, in particular, a complete description of all the periodic conjugacy classes with respect to a given automorphism φ. Our hierarchical decomposition combined with the properties of generalized Dehn extensions and irreducible chambers and their periodic subfactors presented in sections 2 and 4, allows one to analyze all the periodic conjugacy classes of periodic free factors with respect to φ. We conclude this section by showing how a periodic factor inherits its hierarchical decomposition from that of the ambient group F_n.

Theorem 5.7 *With the notation above, let P be a periodic factor with respect to an automorphism φ. Then the blocks in the hierarchical decomposition of P correspond to conjugacy classes of intersections between the periodic factor P and blocks in the hierarchical decomposition of the ambient group F_n.*

Proof: We prove the theorem by induction on the level of huts and blocks in the hierarchical decomposition of the periodic factor P. All huts of level 1 in P are irreducible factors so they are also huts of level 1 in the hierarchical decomposition of the ambient group F_n. Since blocks are obtained from huts by unifying periodic factors, all blocks of level 1 in P are conjugacy classes of intersections between P and blocks of level 1 in the hierarchical decomposition of F_n, so we proved the inductive hypothesis for periodic factors of level 1.

Suppose the blocks of level ℓ in P, $\{BL(P)_j^\ell\}$, are conjugacy classes of intersections between P and blocks of level ℓ in F_n, $\{BL_j^\ell\}$. Since $BL(P)_j^\ell$ is the intersection between the periodic factor P and some conjugate of the block $BL_{j'}^\ell$ for some j', lemma 2.6 implies that the generalized Dehn closure in P of $BL(P)_j^\ell$ is the intersection between P and the generalized Dehn closure (in F_n)of the same conjugate of $BL_{j'}^\ell$, and all conjugacy classes of intersections between the periodic factor P and generalized Dehn closures of the blocks of level ℓ in F_n, $\{GD(BL_j^\ell)\}$, are generalized Dehn closures in P of blocks of level ℓ in P, $\{GD_P(BL(P)_j^\ell)\}$.

By proposition 5.6 every irreducible extension in F_n is conjugate to an irreducible extension A_m^ℓ used in defining one of the huts of some level ℓ in the hierarchical decomposition of F_n. By our inductive hypothesis all irreducible extensions used in defining huts of level at most ℓ in P are irreducible extensions used in defining huts of the same level in F_n, and every irreducible extension in F_n that can be conjugated into P and that is used in defining a hut of level at most ℓ in F_n, is also used in defining a hut of the same level in P. Hence, our inductive hypothesis implies that every irreducible extension A_m^ℓ used in defining a hut of level $\ell+1$ in F_n that can be conjugated into P, is used in defining a hut of level $\ell+1$ in P, and every irreducible extension used in defining a hut of level $\ell+1$ in P is some irreducible extension A^{ℓ}_m used in defining a hut of level $\ell+1$ in F_n.

Since generalized Dehn closures of blocks of level ℓ in P are conjugacy classes of intersections between P and generalized Dehn closures of blocks of level ℓ in F_n, and irreducible extensions used in defining huts of level $\ell+1$ in P are all irreducible extensions used in defining huts of level $\ell+1$ in the ambient group F_n, every hut of level $\ell+1$ in the hierarchical decomposition of P can be conjugated into a hut of level $\ell+1$ in the hierarchical decomposition of F_n. Since blocks are obtained from huts by unifying periodic factors, blocks of level $\ell+1$ in P are conjugacy classes of intersections between P and blocks of level $\ell+1$ in F_n which concludes the proof of the theorem.

$\square$

We have defined the *level of the periodic factor* P, $\ell(P)$, as the minimal level ℓ for which P is a subfactor of a block of level ℓ in the hierarchical decomposition of the ambient group F_n. Since P is a periodic factor, a fixed power of the automorphism φ fixes the conjugacy class of P, so by composing this fixed power with an inner automorphism one obtains a hierarchical decomposition for P associated with it. Theorem 5.7 shows, in particular, that the level of this fixed power of φ restricted to the periodic factor P is identical with the level of P as a periodic factor in F_n.

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