UNIVERSITY OF CALIFORNIA

Los Angeles

Non-Uniquely Ergodic Foliations of Thin Type, Measured Currents and Automorphisms of Free Groups

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Mathematics

by

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1995

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1995

DEDICATION

Diese Arbeit ist meiner geliebten Frau Daniela gewidmet.

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ACKNOWLEDGMENTS

I would like to express my gratitude to my advisor, Mladen Bestvina, for his guidance, support and patience, and for giving me access to unpublished manuscripts. I am very grateful to Mark Feighn, Damien Gaboriau and Frédéric Paulin for making valuable comments on a preliminary draft of the first part of this thesis. For their interest I would like to thank Geoffrey Mess and the members of my doctoral committee Robert Brown, Robert Edwards and Stott Parker. Also, I am thankful to Steve Gersten and Francis Bonahon for inviting me to talk at their seminars. Finally, I would like to thank John Garnett for guidance and encouragement during my early days at UCLA.

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R. Martin (1990), *Ideale der Nevanlinnaklasse,* presented at the annual meeting of the German Mathematical Society, Regensburg, Germany.

R. Martin (1994), *Non-ergodic foliations of thin type,* presented at the Spring Wasatch Topology Conference, Park City.

R. Martin (1994), *Nichtergodische Blätterungen vom dünnen Typ,* presented at the annual meeting of the German Mathematical Society, Duisburg, Germany.

ABSTRACT OF THE DISSERTATION

Non-Uniquely Ergodic Foliations of Thin Type,

Measured Currents and Automorphisms of Free Groups

by

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Doctor of Philosophy in Mathematics

University of California, Los Angeles, 1995

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The characterization of finitely generated groups which can act freely on some $\mathbb{R}$-tree was completed by E. Rips in 1991. The main ingredient of his proof is a classification of minimal foliations on 2-complexes into three types, the toral type, the surface type and the thin type. Examples of non-uniquely ergodic foliations of surface type are known. In the first part of this thesis we construct the first example of a foliation of thin type which is not uniquely ergodic.

The second part is concerned with measured currents for finitely generated free groups, and is based on ideas of M. Bestvina. These currents provide a tool for studying automorphisms of free groups, similar in spirit to Thurston's study of surface automorphisms via his compactification of the Teichmüller space. We first

discuss basic properties of measured currents, then we use them to give a new compactification of the Culler-Vogtmann outer space, and finally we study the action of certain automorphisms on the space of projectivized measured currents.

Part I

Non-Uniquely Ergodic Foliations of Thin Type

CHAPTER 1

Introduction

In 1963 Lyndon ([18]) asked (using the language of length functions) for the classification of finitely generated groups which can act freely on some $\mathbb{R}$-tree by isometries. (An $\mathbb{R}$-tree is an arcwise connected metric space in which every embedded arc is isometric to an interval of $\mathbb{R}$.) Morgan and Shalen conjectured in the 1980s that such a group has to be the free production of free abelian groups and fundamental groups of surfaces.

The breakthrough came with Rips' work in 1991, giving an affirmative answer to the Morgan-Shalen conjecture. Accounts of this proof are given by Bestvina and Feighn [3] and by Gaboriau, Levitt and Paulin [12].

The critical ingredients of the proof are foliated 2-complexes. Their minimal components can be classified into three types, the *toral* type, the *surface* type and the *thin* type, using the terminology of Bestvina and Feighn. The terminology of Gaboriau, Levitt and Paulin calls these types *homogeneous* type, *interval exchange* type and *Levitt* type, respectively.

Foliations of surface type essentially correspond to interval exchange transformations. In 1975 Keane [14] conjectured that all minimal interval exchange transformations are uniquely ergodic, i.e., allow only one transversal measure up to scale. But counterexamples were found shortly thereafter by Keynes and Newton [17] as well as by Keane himself [15]. Veech [25] and Mazur [20] showed that almost all minimal interval exchange transformations are uniquely ergodic (provided the associated permutation satisfies an obviously necessary condition). See also Rees [23] and Kerckhoff [16] for the same result.

On the other hand, toral components are easily seen to be always uniquely ergodic.

Thus a very natural question is whether non-unique ergodicity can occur in the thin case. The main objective of the first part of this thesis is to give an affirmative answer by constructing an example of thin type which is not uniquely ergodic.

CHAPTER 2

Systems of Isometries of Thin Type

2.1 Systems of Isometries

Let D be a disjoint union of finitely many compact (nondegenerate) intervals of $\mathbb{R}$. A *system of isometries* X on $D_X = D$ is a finite collection of isometries $\varphi_i : A_i \to B_i$ $(i = 1, \ldots, N)$ between closed subintervals A_i and B_i of D_X. These $2N$ intervals are called *bases*. For convenience we will often refer to systems of isometries just as *systems*.

An X-*orbit* is an equivalence class of the equivalence relation on D_X generated by the relations $a \sim \varphi_i(a)$ for $a \in A_i$. In other words, x and $y \in D_X$ are in the same X-orbit if and only if we can write

$$y = \varphi_{i_n}^{\varepsilon_n} \circ \cdots \circ \varphi_{i_1}^{\varepsilon_1}(x),$$

where $i_1, \ldots, i_n \in \{1, \ldots, N\}$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$, and the expression is well-defined, in the sense that

$$\varphi_{i_{k-1}}^{\varepsilon_{k-1}} \circ \cdots \circ \varphi_{i_1}^{\varepsilon_1}(x) \in \begin{cases} A_{i_k} & \text{if } \varepsilon_k = 1 \\ B_{i_k} & \text{if } \varepsilon_k = -1 \end{cases}$$

for all $k = 1, \ldots, n$.

An X-orbit is called *singular* if it meets the boundary ∂D_X of D_X or the boundary of some base. Otherwise, it is called *regular.* So, there are only a finite number of singular orbits.

To a system X we can associate a foliated 2-complex ΣX, the *suspension* of X, as follows. For each isometry $\varphi_i : A_i \to B_i$ consider (a copy of) $A_i \times [0,1]$, which we call a *band,* foliated by the sets $\{a\} \times [0,1]$ for $a \in A_i$. Now attach $A_i \times [0,1]$ to D_X by identifying each $\{a\} \times \{0\} \subset A_i \times [0,1]$ with $\{a\} \subset D_X$ and each $\{a\} \times \{1\} \subset A_i \times [0,1]$ with $\{\varphi_i(a)\} \subset D_X$.

The foliation on ΣX is induced from the foliations of the bands. This means that x and $y \in \Sigma X$ are in the same leaf (called an X-*leaf*) if and only if there is a sequence $\ell_1, \ldots, \ell_n$ of leaves of bands with $x \in \ell_1$, $y \in \ell_n$ and $\ell_{i-1} \cap \ell_i \neq \emptyset$ for $i = 2, \ldots, n$. So, the intersections of leaves of ΣX with D_X are precisely the X-orbits. We will often abuse language and not distinguish between systems of isometries and their suspensions. In particular, we will often call an isometry φ_i a band. Also, the *width* of a band or an isometry is the length of any of its bases.

Two systems X and X' are *homeomorphic (as systems of isometries)* if they have the same number of isometries and if there exists a homeomorphism $f : D_X \to D_{X'}$ which respects all the isometries, in the sense that $f(A_i) = A'_i$ and that $f \circ \varphi_i = \varphi'_i \circ f$ on A_i for all i. (Here, primes refer to concepts corresponding to X'.)

We call a system X *uniquely ergodic* if any X' which is homeomorphic to X is already homeomorphic to X via a local isometry up to (global) rescaling.

2.2 Minimal Systems and Systems of Thin Type

A system X is called *minimal* if each X-orbit is dense in D_X. Any system of isometries can be decomposed into minimal components and finite orbits in a certain way. So, it is natural to concentrate on minimal systems. We do not have a need for the details of this decomposition, which is independently due to Morgan and Shalen and to Imanishi. See [3] or [12] for precise statements.

Rips classified minimal systems of isometries into three types, the *toral* type, the *surface* type, and the *thin* type. To give precise definitions, one needs the so-called *Rips' machine.* Full details can be found in [3]. For our purposes it will suffice to limit ourselves to a restricted class of thin systems. But these systems are in most ways representative for all systems of thin type.

Given a system X, consider the set of points in D_X which belong to exactly one base. The components of this set are intervals, called *free arcs.* If X has free arcs, let I be one of them. Then I is contained in a single base, say A_i, and I is relative open in A_i. Now replace φ_i by its restrictions to the nondegenerate components of $A_i - I$, to get a new system X' on $D_{X'} = \text{clos}(D_X - \text{clos}(I))$. We say we get X' by *collapsing* from the free arc I. If X is minimal, then so is X'.

If we have two bases of distinct bands, say B_i and A_j with $i \neq j$, which coincide ($B_i = A_j$) and form a component of D which does not meet any other bands, we can replace φ_i and φ_j by their composition $\varphi_j \circ \varphi_i : A_i \to B_j$ and replace D by

$D - B_i$. We call this operation *replacing a long band by a single band.* Typically, collapsing from a free arc will generate a long band, and we will tacitly replace it by a single band.

If X is a minimal system on which we can perform an infinite sequence of collapses, then X is of thin type.

2.3 The Basic Example

To understand the non-uniquely ergodic example we are going to present, it is useful to review the basic example of a system of thin type (cf. [3]).

Let $D_X = [0, 1]$. We have three (orientation preserving) isometries

$$\begin{aligned} \varphi_a : &\quad [0, \lambda^4] \to [1 - \lambda^4, 1], \\ \varphi_b : &\quad [0, \lambda^3] \to [\lambda - \lambda^4, \lambda], \\ \varphi_c : &\quad [0, \lambda^2] \to [1 - \lambda^2, 1]. \end{aligned}$$

The number λ is the unique positive root of $\lambda^3 = 1 - \lambda$.

Figure 2.1 shows a schematic representation of this example X. Two lines on the same vertical level indicate the two bases of an isometry.

The arc $(\lambda, 1 - \lambda^4)$ is free. If we collapse it, we get the band complex which is shown in Figure 2.2. Note that the length of the shortest base is $\lambda^2 + \lambda - 1 = \lambda^2 - \lambda^3 = \lambda^5$.

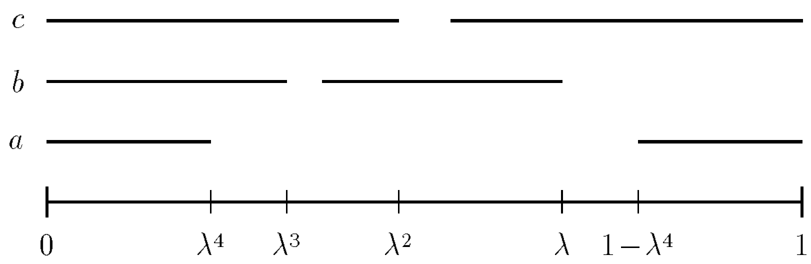

Figure 2.1: The basic example of a system of thin type

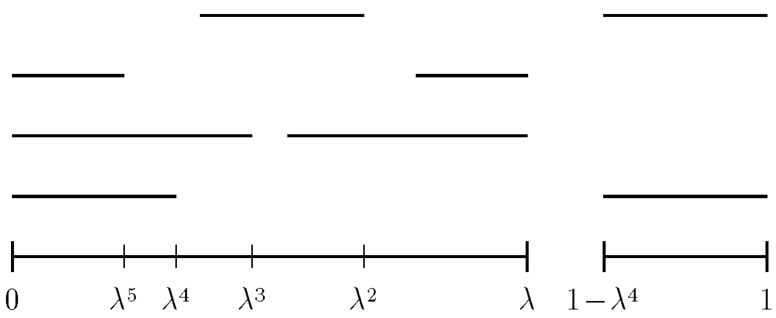

Figure 2.2: The basic example after collapsing a free arc

Now we replace the long band by a single band, with the result X' as shown in Figure 2.3.

But this system X' is just the original system X, scaled down by a factor of λ and with the rôle of the bands changed. In other words, the system we get after collapsing the free arc and replacing the long band by a single band is homeomorphic to the original band complex. Thus it is obvious that we can perform an infinite sequence of collapses on X. By the same methods we will use for our non-uniquely ergodic example it follows that X is minimal. Thus, X is indeed of thin type.

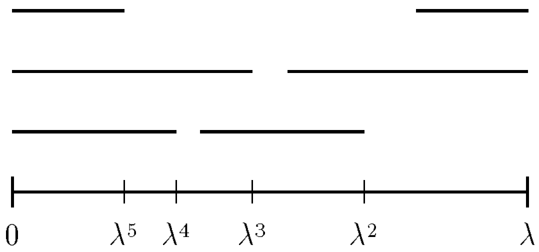

Figure 2.3: The result after replacing the long band by a single band

2.4 Further Examples

We will now present two ideas with which to generalize the basic examples, both of which will be used for our non-uniquely ergodic example.

2.4.1 Using More Bands

The first idea is to use n bands in the following way, where n is some integer at least 3. Let λ be the (unique) positive solution to $\lambda^n = 1 - \lambda$.

Again, $D_X = [0, 1]$. All isometries are orientation preserving. There is one isometry

$$[0, \lambda^{2n-2}] \to [1 - \lambda^{2n-2}, 1],$$

and for $i = 1, \ldots, n-1$ there is an isometry

$$[0, \lambda^{n-2+i}] \to [\lambda^{i-1} - \lambda^{n-2+i}, \lambda^{i-1}].$$

For $n = 3$ we just get the basic example. In Figure 2.4 we show the case with $n = 5$.

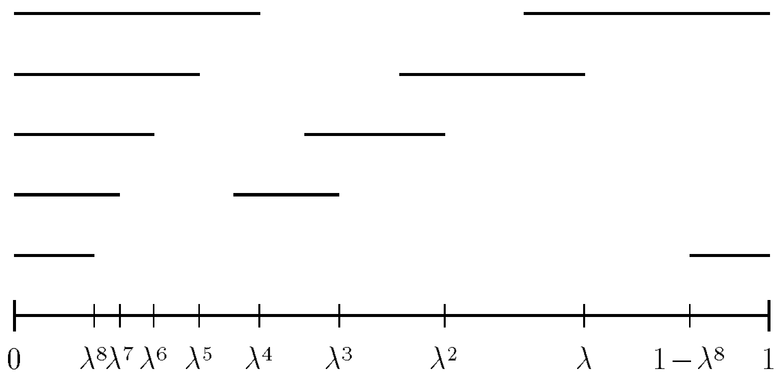

Figure 2.4: A thin example with five bands

Similar as in the basic example, collapsing the free arc $(\lambda, 1-\lambda^{2n-2})$ and replacing the long band by a single band yields the system we started with, but scaled down by a factor of λ. Again, we get a system of thin type.

2.4.2 Shortenings

Now we introduce *shortenings,* which will play a key rôle later.

Consider the system X on $[0,1]$ with the three isometries

$$\left[0, \frac{\lambda^2(\lambda^2+n)}{n+1}\right] \to \left[1 - \frac{\lambda^2(\lambda^2+n)}{n+1}, 1\right],$$
$$\left[0, \frac{\lambda(\lambda^2+n)}{n+1}\right] \to \left[\lambda - \frac{\lambda(\lambda^2+n)}{n+1}, \lambda\right],$$
$$\left[0, \frac{\lambda^2+n}{n+1}\right] \to \left[1 - \frac{\lambda^2+n}{n+1}, 1\right].$$

This time, the number λ is the unique positive root of $\lambda^3 = 1 - (n+1)\lambda$, for some integer $n \geq 0$.

Consider the intervals

$$I_k = \left(\frac{\lambda^2 + n}{n+1} - k \cdot \frac{1-\lambda^2}{n+1}, 1 - \frac{\lambda^2(\lambda^2+n)}{n+1} - k \cdot \frac{1-\lambda^2}{n+1} \right)$$

for $k = 0, \ldots, n-1$. We can collapse the free arc I_0, and we get a system of isometries similar to X, except that the longest band is shortened by $(1-\lambda^2)/(n+1)$. We call such a step informally a *shortening.* Now the result of the shortening has I_1 as a free arc, and we can again perform a shortening. Continuing this process, we perform $n-1$ shortenings by collapsing the arcs $I_0, I_1, \ldots, I_{n-1}$. Finally, we can perform a shuffling by collapsing the free arc $(\lambda, (1-\lambda^4)/(n+1))$, and a straightforward calculation shows that the resulting system is essentially X, but scaled down by a factor of λ. So, as before, we can perform an infinite sequence of collapses, and X is of thin type.

Let us illustrate our example in the case $n = 2$ with some illustrations. The original system X is shown in Figure 2.5. Here we have some bands whose two bases overlap. We draw those two bases on two slightly different vertical levels.

The results after one and after two shortenings are shown in Figures 2.6 and 2.7.

Performing one more collapse, i.e., a shuffling, yields the system in Figure 2.8. And indeed, this system is homeomorphic to our original system X, via scaling down by a factor of λ.

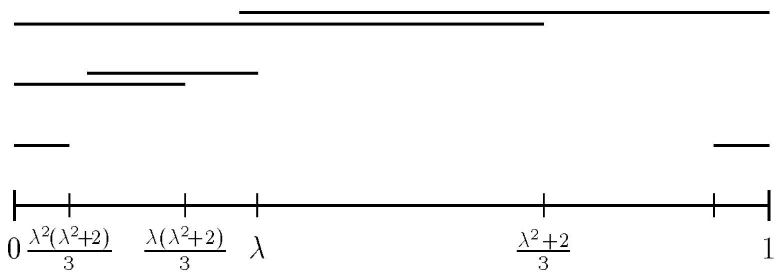

Figure 2.5: An example with shortenings

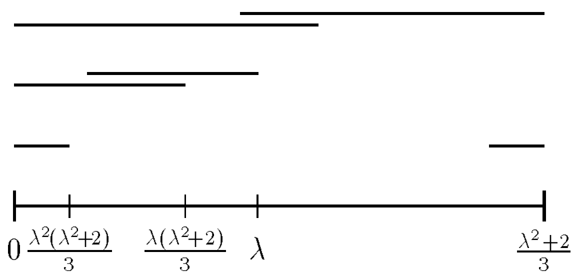

Figure 2.6: The example after the first shortening

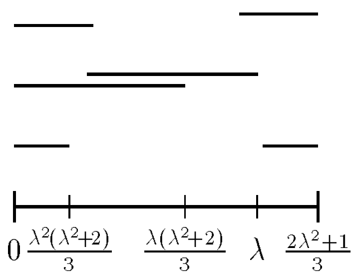

Figure 2.7: The example after the second shortening

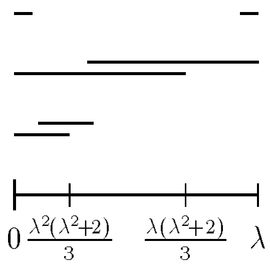

Figure 2.8: The example after the shuffle

CHAPTER 3

An Example of a Non-Uniquely Ergodic Thin Foliation

3.1 The Construction

We will use two disjoint copies of the real numbers, denoted $\mathbb{R}_0$ and $\mathbb{R}_1$, writing for example $[0,1]_1$ for an interval of $\mathbb{R}_1$. Also, we regard the subscripts 0 and 1 as the two elements of the group $\mathbb{Z}/2\mathbb{Z}$.

Let V be the vector space over $\mathbb{Z}/2\mathbb{Z}$ of dimension five, with the basis $S = \{a,b,c,d,e\}$. For $v \in V$ and $x \in S$, let $v_x \in \mathbb{Z}/2\mathbb{Z}$ denote the x-coordinate of v. Thus, $v = (v_a, v_b, v_c, v_d, v_e)$.

We now define systems of isometries X of *type* K_v, where $v \in V$. They depend on positive parameters $(\ell_a^0, \ldots, \ell_e^0, p_b^0, \ldots, p_e^0, \ell_a^1, \ldots, \ell_e^1, p_b^1, \ldots, p_e^1) \in \mathbb{R}^{18}$ satisfying the inequalities $\ell_x^\varepsilon + \ell_y^{\varepsilon+v_y} < p_y^\varepsilon \leq p_e^\varepsilon$ and $\ell_z^\varepsilon + \ell_a^{\varepsilon+v_a} < p_e^\varepsilon$, where $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ and $x,y,z \in S$ with $x < y$, using the lexicographic order. The subset of $\mathbb{R}^{18}$ of these parameters forms a cone and is denoted by Δ_v.

Let $D = D_X$ be the union of the two intervals $[0, p_e^0]_0$ and $[0, p_e^1]_1$. The (orientation preserving) isometries of X are $[0, \ell_a^\varepsilon]_\varepsilon \to [p_e^{\varepsilon+v_a} - \ell_a, p_e^{\varepsilon+v_a}]_{\varepsilon+v_a}$ and $[0, \ell_x^\varepsilon]_\varepsilon \to [p_x^{\varepsilon+v_x} - \ell_x, p_x^{\varepsilon+v_x}]_{\varepsilon+v_x}$ for $x \in S - \{a\}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$.

So, these systems consist of ten isometries. Each component of D is the domain of five isometries, which we think of as labeled by a, b, c, d and e. The domain and image of an isometry labeled x are in the same component if and only if $v_x = 0$. In particular, ΣX_v is connected except when $v = 0$. See Figure 3.1 for an example of type K_{a+c+d}.

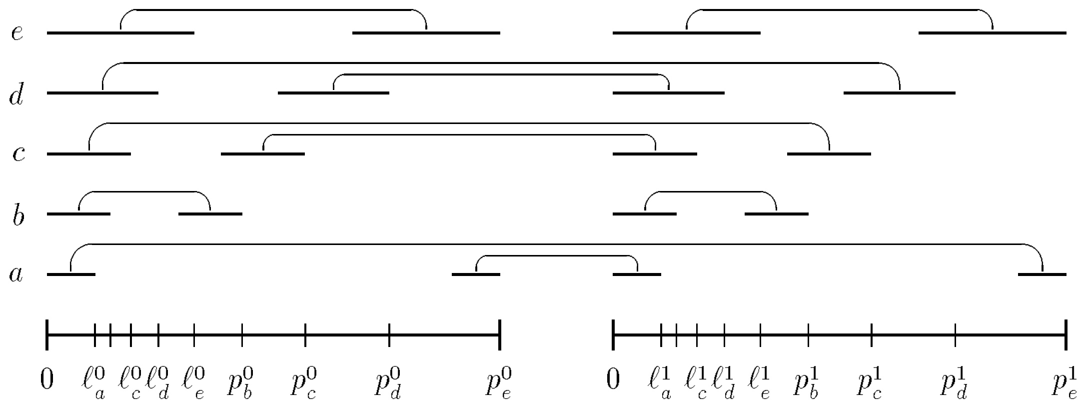

Figure 3.1: A system of isometries of type K_{a+c+d}

Now consider a system of isometries X of type K_v whose parameters additionally satisfy the inequalities $p_b^\varepsilon, p_c^\varepsilon, \ell_e^\varepsilon, p_e^\varepsilon - \ell_e^{\varepsilon+v_e} < p_d^\varepsilon < p_e^\varepsilon - \ell_a^{\varepsilon+v_a}$ for $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Then $[p_d^\varepsilon, p_e^\varepsilon - \ell_a^{\varepsilon+v_a}]_\varepsilon$ is a free arc and can be collapsed (for both values of ε). Replacing the two long bands just created by two single bands gives a system X' essentially of type $K_{\psi v}$ (not all inequalities between the parameters are necessarily satisfied), where ψ is the automorphism of V given by $\psi a = b, \psi b = c, \psi c = d, \psi d = e$ and $\psi e = a + b$. We call this operation on X a *shuffle,* since it resembles the operation we used in the last chapter.

The new parameters $(\ell_a^0, \ldots, p_e^1)$ can be calculated from the original parameters via the endomorphism of $\mathbb{R}^{18}$ given by the 18×18-matrix M_v, where

$$M_v = \begin{cases} \begin{pmatrix} M' + M'' & 0 \\ & \\ 0 & M' + M'' \end{pmatrix} & \text{if } v_e = 0, \\ \begin{pmatrix} M' & M'' \\ & \\ M'' & M' \end{pmatrix} & \text{if } v_e = 1, \end{cases}$$

with

$$M' = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$M'' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly, if $v_e = 0$ and the inequalities $p_b^\varepsilon, p_c^\varepsilon, p_d^\varepsilon, p_e^\varepsilon - \ell_e^{\varepsilon+v_e} < \ell_e^\varepsilon < p_e^\varepsilon - \ell_a^{\varepsilon+v_a}$ hold for $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, then $[\ell_e^\varepsilon, p_e^\varepsilon - \ell_a^{\varepsilon+v_a}]_\varepsilon$ is a free arc and can be collapsed, and we get a system which is essentially of type K_v. We call this operation a *shortening,* in analogy to the operation used in the last chapter.

The parameter change for a shortening can be described by the 18×18-matrix

$$N_v = \begin{pmatrix} N' & 0 \\ 0 & N' \end{pmatrix}$$

with

$$N' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is straightforward but tiresome to verify the inclusion $M_{\psi^{-1}v}(\Delta_v) \subset \Delta_{\psi^{-1}v}$ and for $v_e = 0$ the inclusion $N_v(\Delta_v) \subset \Delta_v$. So, parameters in $M_v^{-1}(\Delta_v)$ correspond to systems of type $K_{\psi^{-1}v}$ on which we can perform a shuffling and get a system of type K_v. Similarly, parameters in $N_v^{-1}(\Delta_v)$ with $v_e = 0$ correspond to systems of type K_v on which we can perform a shortening and get a system of type K_v.

For our construction we are going to make use of the ψ-orbit

$$a + c + d \longmapsto b + d + e \longmapsto a + b + c + e \longmapsto a + c + d$$

of length 3. This corresponds to a sequence of systems as shown in Figure 3.2.

Consider the matrix

$$\begin{aligned} A_k &= (M_{a+b+c+e} M_{b+d+e} M_{a+c+d} N^k_{a+c+d})^{-1} \\ &= \begin{pmatrix} A'_k & A''_k \\ A''_k & A'_k \end{pmatrix}, \end{aligned}$$

where

$$A'_k = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -k & 0 & 0 & 0 & 0 & 1 & k \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1-k & 0 & 0 & 0 & 0 & 1 & 1+k \end{pmatrix}$$

and

$$A''_k = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -k & -k & 0 & 0 & 0 & k & k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1-k & -1-k & 0 & 0 & 0 & 1+k & 1+k & 0 & 0 \end{pmatrix}.$$

So A_k/k converges with $k \to \infty$ (elementwise) to the matrix

$$A = \begin{pmatrix} A' & A'' \\ A'' & A' \end{pmatrix}$$

with

$$A' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$A'' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Note that $A'^2 + A''^2 = A'$ and $A'A'' + A''A' = A''$, and so A is idempotent. Also note that A is of rank two. These two facts will play an essential rôle later, when we make our example non-uniquely ergodic.

Given a system of isometries X of type K_{a+c+d}, the transformation corresponding to A_k yields a system of type K_{a+c+d} on which we can perform k shortenings and then three shuffles to get X. Let us call these steps together a *k-step.*

For the following we need a few more definitions. An *X°-orbit* is is an equivalence class of the equivalence relation on D_X generated by the relations $a \sim \varphi_i(a)$ for $a \in A_i^\circ$, where A_i° is A_i with the two boundary points removed. So, each X°-orbit is contained in an X-orbit. Again, an X°-orbit is *singular* if it contains any boundary point of D_X or of some base, and *regular* otherwise. Thus, the regular X°-orbits are precisely the regular X-orbits. Also, we will talk about *X°-leaves.*

We will also call orbits consisting of only one point *trivial* orbits.

Proposition 1 *Given any sequence $k_1, k_2, \ldots$ of nonnegative integers, there exists a minimal thin system of isometries of type K_{a+c+d} on which one can perform* *first a k_1-step, then a k_2-step, and so on. In other words, there is a sequence of minimal systems $X_0, X_1, X_2, \ldots$ of type K_{a+c+d} such that for each n a k_n-step applied to X_{n-1} yields X_n.*

Proof. Let $B_n = A_{k_1} A_{k_2} \cdots A_{k_n}$ and define

$$\Delta = \bigcap_{n=1}^{\infty} B_n(\mathrm{clos}(\Delta_{a+c+d})) - \{0\}.$$

Then

$$\mathbb{P}\Delta = \bigcap_{n=1}^{\infty} B_n(\mathrm{clos}(\mathbb{P}\Delta_{a+c+d})),$$

where $\mathbb{P}$ denotes the natural projection from $\mathbb{R}^{18} - \{0\}$ to the projective space $\mathbb{RP}^{17}$. Since $A_k^{-1}(\Delta_{a+c+d}) \subset \Delta_{a+c+d}$, the set $\mathbb{P}\Delta$ is the intersection of a decreasing sequence of nonempty compact sets, and thus nonempty. Therefore, the cone Δ is also nonempty.

It is easy to check that $\Delta \subset \Delta_{a+c+d}$ holds, and thus we actually have $\Delta = \bigcup B_n(\Delta_{a+c+d})$. So, if we pick $(\ell^0_a, \ldots, p^1_e) \in \Delta$, these parameters describe a system X_0 of type K_{a+c+d} on which we can perform the steps as required.

It remains to establish that $X = X_0$ is indeed minimal. By the decomposition theorem due to Imanishi (see [12], Theorem 3.1) and to Morgan and Shalen, it is enough to show that there are no finite X°-orbits in $D_X - \partial D_X$ and that there does exist some infinite X-orbit. Here we use the fact that X has isometries whose two bases are in distinct components of D_X (since $a + c + d \neq 0$), or, in other words, that there is some X°-orbit which meets each component of D_X.

Consider the sequence of systems $X = X_0 \rightsquigarrow X_1 \rightsquigarrow X_2 \rightsquigarrow \cdots$ as above, but without simplifying long bands to single bands. So X_n is a "subsystem" of X, and any X_n-orbit is contained in an X-orbit. The *length* of the long bands (i.e., the number of single bands they are composed of) grows arbitrarily large. This follows from the fact that the 10×10-matrix describing the change of lengths of the ten bands under a k-step is elementwise bounded below by the matrix describing the change of lengths under a 0-step, and this later matrix is nonnegative and irreducible.

The length of the long bands being unbounded has several consequences. The first consequence is that the X-orbit containing the left boundary point of $[0, p_e^0]_0 \subset D_X$ is infinite.

The second consequence is that during the process we have $p_e^0 \to 0$ and $p_e^1 \to 0$. The reason is that the length times the width of the bands is clearly bounded above, so the width of the bands has to tend to zero.

This also yields that every point of $D_{X_n} - \partial D_{X_n}$ is contained in the interior of some base. So the only trivial X_n°-orbits are those in ∂D_{X_n}.

The only possible effect of a collapse on a nontrivial X°-leaf is to decompose it into a collection of smaller orbits, and it is easy to check that in our situation not all of them can be trivial. Also, each X_n°-orbit has at least as many elements as the length of one of the long bands. Therefore, by choosing n so large that the length of each long band is longer that the size of the largest finite X°-orbit, we see that each X°-orbit must be trivial, and thus contained in ∂D_X. As noted above, this implies the minimality of X. $\square$

3.2 Choosing the Parameter Sequence

As noted in the proof of Proposition 1, the elements of

$$\mathbb{P}\Delta = \bigcap_{n=1}^{\infty} B_n(\mathbb{P}\Delta_{a+c+d})$$

describe (up to scale) systems of isometries of thin type. Two elements of $\mathbb{P}\Delta$ describe two systems which are homeomorphic as systems. The homeomorphism can be constructed by defining it first on the (dense) X-orbit containing the left-hand boundary point of $[0, p_e^0]_0 \subset D_X$, and then extending it to D_X by continuity.

On the other hand, two distinct elements of $\mathbb{P}\Delta$ give rise to systems which are obviously not homeomorphic via a local isometry up to rescaling. (To avoid clumsy statements we ignores the trivial situation where we have an local isometry which interchanges the two components of D_X.)

This yields the following.

Proposition 2 *If we choose the sequence $k_1, k_2, \ldots$ of nonnegative integers in such a way that $\mathbb{P}\Delta$ consists of more than one element, each of these elements yields a non-uniquely ergodic minimal system of thin type.*

So our goal is to show that such a choice is indeed possible. For that we are going to use a lemma which holds in a quite general context.

Recall that a *semigroup* G is a set with an associative multiplication $G \times G \to G$. An element $x \in G$ is *idempotent* if $x \cdot x = x$. A *metric semigroup* is a semigroup with a metric for which the multiplication is continuous.

Lemma 3 *Let $x_1, x_2, \ldots$ be a sequence in a metric semigroup G (with metric d) converging to an idempotent element x, and let $\varepsilon > 0$. Then there exist a sequence of positive integers $k_1, k_2, \ldots$ such that $d(x_{k_1} x_{k_2} \cdots x_{k_n}, x) < \varepsilon$ for all n.*

Proof. Write $y_n = x_{k_1} \cdots x_{k_n}$. We use induction to choose $k_1, k_2, \ldots$ such that both $d(y_n, x) < \varepsilon(1 - 1/2^n)$ and $d(y_n x, x) < \varepsilon(1 - 1/2^n)$ hold for all $n \geq 1$.

For the induction step use continuity of multiplication to choose k_n such that $d(y_{n-1} x_{k_n}, y_{n-1} x) < \varepsilon / 2^n$ and

$$d(y_{n-1} x_{k_n} x, y_{n-1} x) = d(y_{n-1} x_{k_n} x, y_{n-1} x x) < \frac{\varepsilon}{2^n}.$$

Then we get

$$\begin{aligned} d(y_n, x) &\leq d(y_{n-1} x_{k_n}, y_{n-1} x) + d(y_{n-1} x, x) \\ &< \frac{\varepsilon}{2^n} + \varepsilon \left(1 - \frac{1}{2^{n-1}}\right) = \varepsilon \left(1 - \frac{1}{2^n}\right) \end{aligned}$$

and

$$d(y_n x, x) \leq d(y_{n-1} x_{k_n} x, y_{n-1} x) + d(y_{n-1} x, x) < \varepsilon \left(1 - \frac{1}{2^n}\right). \quad \square$$

Proposition 4 *It is possible to choose the sequence $k_1, k_2, \ldots$ such that the set $\mathbb{P}\Delta$ consists of more than one point.*

Proof. Fix any norm on $\mathbb{R}^{18}$ and let Δ'_{a+c+d} be the set of elements in Δ_{a+c+d} of norm 1. Since Δ_{a+c+d} contains a simplex of dimension 18, and since the rank of the matrix $A = \lim A_k/k$ is 2, the set $A(\Delta'_{a+c+d})$ contains two linear independent elements y_1 and y_2. Pick $x_i \in \Delta'_{a+c+d}$ with $y_i = Ax_i$.

Choose $\varepsilon > 0$ small enough such that the projections of the ε-neighborhoods of y_1 and y_2 to $\mathbb{RP}^{17}$ have a distance bounded away from zero, in some metric on $\mathbb{RP}^{17}$.

Applying Lemma 3 to the multiplicative semigroup of all 18×18-matrices and to the sequence A_k/k converging to A, we get a sequence $k_1, k_2, \ldots$ such that $B_n(x_i)$ is in the ε-neighborhood of y_i. Thus, the set $B_n(\mathbb{P}\Delta_{a+c+d}) \subset \mathbb{RP}^{17}$ has a diameter bounded away from zero, and therefore the intersection $\mathbb{P}\Delta$ of this decreasing sequence of compact sets has a positive diameter. In other words, $\mathbb{P}\Delta$ consists of more than one point. $\square$

Finally, Proposition 2 and Proposition 4 together yield the desired result.

Theorem 5 *There do exist minimal systems of isometries of thin type which are not uniquely ergodic.*

We can rephrase this result in the language of group actions on $\mathbb{R}$-trees. The fundamental group of a minimal thin band complex X whose leaves have no loops acts on the $\mathbb{R}$-tree given by the leaf space of the universal cover of X by a minimal thin action.

Corollary 6 *There do exist two minimal thin actions of some finitely generated free group on $\mathbb{R}$-trees, such that there is an equivariant homeomorphism but not an equivariant homothety (isometry up to scale) between the two $\mathbb{R}$-trees.*

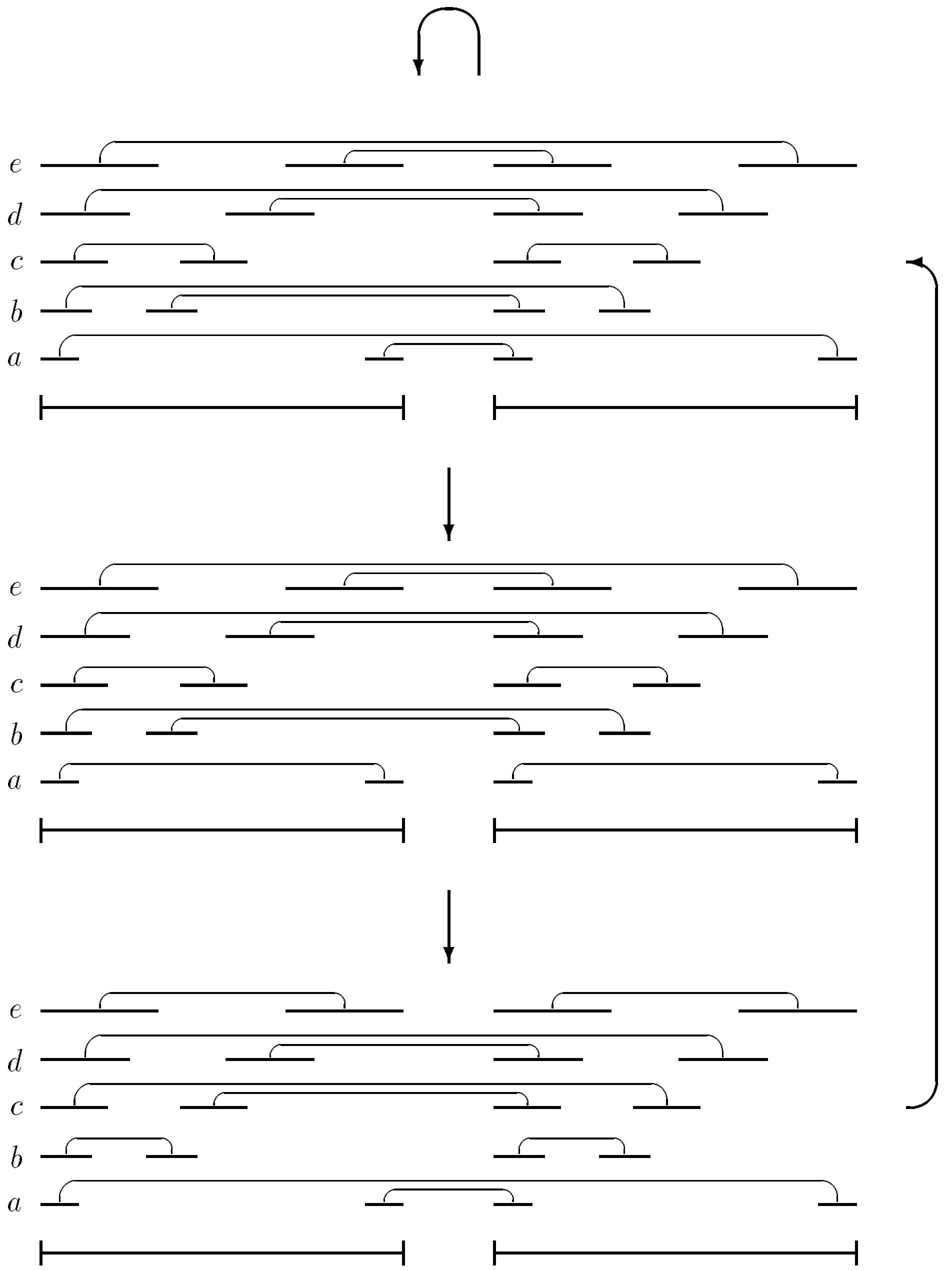

Figure 3.2: The combinatorics of the non-unique ergodic example

Part II

Measured Currents and Free Groups Automorphisms

CHAPTER 4

Introduction

To study mapping classes of a compact surface S, W. Thurston successfully introduced powerful geometric techniques. Using measured laminations, he compactified the Teichmüller space $\mathcal{T}(S)$ of the surface in such a way that the action of the mapping class group on $\mathcal{T}(S)$ extends to this compactification, which is topologically a Euclidian ball.

Motivated by this successful approach, M. Culler and K. Vogtmann [9] attempted to transfer these techniques to the study of automorphisms of a finitely generated free group $\mathbb{F}$. They introduced what nowadays is known as the Culler-Vogtmann outer space CV($\mathbb{F}$), consisting essentially of marked metric graphs, or equivalently, of certain actions of $\mathbb{F}$ on trees. Also, they constructed a compactification of CV($\mathbb{F}$) using length functions. The group of outer automorphisms Out($\mathbb{F}$) acts on CV($\mathbb{F}$) and its compactification.

But this compactified outer space is topologically more complex than the compactified Teichmüller space. The theory turns out to be less satisfying than in the surface case. This motivates the search for alternative spaces on which Out($\mathbb{F}$) acts.

An idea of M. Bestvina [1] is to consider so-called measured currents. The space of measured currents $\mathcal{MC}(\mathbb{F})$ contains conjugacy classes of $\mathbb{F}$ as a dense subset. It was introduced in the surface setting by F. Bonahon in [7] to give an alternative approach to Thurston's compactification of $\mathcal{T}(S)$. Also, J.-P. Otal [22] studied them in the surface case.

This part of this thesis serves to present M. Bestvina's ideas and to extend on them. The first chapter introduces measured currents and gives some of their basic properties. In the next chapter, we will compactify Culler-Vogtmann outer space using measured currents. Interestingly, as opposed to the situation in the surface case, this construction yields a compactification distinct from the one via length functions. In the final chapter we will study the dynamics of automorphisms acting on $\mathcal{MC}(\mathbb{F})$. Some additional material is treated in the appendices.

CHAPTER 5

Basic Properties of Measured Currents

5.1 Free Groups and Their Automorphisms

The basic object of our study is the free group $\mathbb{F} = \mathbb{F}_n$ on $n > 1$ generators. The number of generators is called the *rank* of $\mathbb{F}$ and denoted $\mathrm{rk}(\mathbb{F})$. For a good reference on free groups see the book by Lyndon and Schupp [19]. In this and the next section we present some basic facts and definitions related to free groups.

The group of automorphisms of $\mathbb{F}$ is denoted by $\mathrm{Aut}(\mathbb{F})$, and the group of inner automorphisms (conjugations) by $\mathrm{Inn}(\mathbb{F})$. Elements of the quotient $\mathrm{Out}(\mathbb{F}) = \mathrm{Aut}(\mathbb{F})/\mathrm{Inn}(\mathbb{F})$ are called *outer automorphisms.*

We denote the standard generators of $\mathbb{F}$ by the letters $a, b, \ldots, c$. The (ordered) set of these generators is called the *standard basis.* Elements of $\mathbb{F}$ correspond to (reduced) words in the letters $a, b, \ldots, c$ and their inverses $\bar{a}, \bar{b}, \ldots, \bar{c}$. The group operation corresponds to concatenation (and reduction) of words. We will often use this correspondence tacitly. The length of the word corresponding to w is denoted by $|w|$.

We denote the product of two words v and w simply as vw, and we sometimes denote this product by $v * w$ if there is no canceling, i.e., if $|vw| = |v| + |w|$. If $w = u_1 * v * u_2$, then we say that v is a *subword* of w. In the special case that $u_1 = 1$, we call v an *initial* subword of w, and we write $v \leq w$. In the other special case $u_2 = 1$, we say v is a *terminal* subword of w.

Any image of the standard basis under an automorphism of $\mathbb{F}$ is called a *basis.* Often we will denote bases of $\mathbb{F}$ by $\mathcal{B}$, and the standard basis by $\mathcal{B}_0$. Any element of $\mathbb{F}$ can also be regarded as a word in the alphabet $\mathcal{B}$. The length of w with respect to a basis $\mathcal{B}$ is denoted $|w|_{\mathcal{B}}$ and is also referred to as $\mathcal{B}$-length. So $|w|_{\mathcal{B}_0} = |w|$ and $|\Phi w|_{\Phi\mathcal{B}} = |w|_{\mathcal{B}}$ for $\Phi \in \mathrm{Aut}(\mathbb{F})$.

We denote the conjugacy class of w in $\mathbb{F}$ by $[w]$, and the set of all conjugacy classes of $\mathbb{F}$ by $[\mathbb{F}]$. Note that $\mathrm{Out}(\mathbb{F})$ acts in a natural way on $[\mathbb{F}]$. With respect to the standard basis, the classes in $[\mathbb{F}]$ correspond to cyclic words (cyclic reduced words modulo conjugation). We let $|\alpha|$ for $\alpha \in [\mathbb{F}]$ to be the length of the cyclic word corresponding to α. So we have $|[w]| = |w|$ if and only if w is cyclically reduced. Similarly, we define $|\alpha|_{\mathcal{B}}$ for any basis $\mathcal{B}$.

For an automorphism $\Phi \in \mathrm{Aut}(\mathbb{F})$ we write $|\Phi|_{\mathcal{B}} = \max_{x \in \mathcal{B}} |\Phi x|_{\mathcal{B}}$. This implies the inequality $|\Phi w|_{\mathcal{B}} \leq |\Phi|_{\mathcal{B}} \cdot |w|_{\mathcal{B}}$.

A *free factor* of $\mathbb{F}$ is a subgroup generated by a subset of some basis of $\mathbb{F}$. Any free factor is a free group of rank not more than $\mathrm{rk}(\mathbb{F})$.

The *width* of $w \in \mathbb{F}$, denoted $\mathrm{width}(w)$, is the rank of the smallest free factor of $\mathbb{F}$ containing w. (Note that the intersection of two free factors of $\mathbb{F}$ is again a free factor. See for example Problem 32 on p.117 in [21].) Equivalently, $\mathrm{width}(w)$ is the smallest number of distinct letters needed to write w as a word in some basis. The *width* of a conjugacy class of $\mathbb{F}$ is the width of any of its elements.

If a nontrivial element or conjugacy class of $\mathbb{F}$ is not a proper power, we call it *indivisible.* Clearly, each nontrivial element is the power of some indivisible element.

5.2 The Boundary of Free Groups

Given a free group $\mathbb{F}$ with basis $\mathcal{B}$, we can look at its Cayley graph, denoted by $\mathrm{Cayley}(\mathbb{F}, \mathcal{B})$. It is a simplicial tree where each vertex has valence $2\,\mathrm{rk}(\mathbb{F})$. The space of ends of $\mathrm{Cayley}(\mathbb{F}, \mathcal{B})$ compactifies the discrete space $\mathbb{F}$ and is called the boundary $\partial\mathbb{F}$ of $\mathbb{F}$. It is homeomorphic to the Cantor set. The Bounded Cancellation Proposition (Appendix A, Proposition 46) implies that the construction of $\partial\mathbb{F}$ does not depend on the particular basis $\mathcal{B}$ chosen. In other words, any automorphism of $\mathbb{F}$ induces a homeomorphism of $\partial\mathbb{F}$.

Elements of $\partial\mathbb{F}$ can be regarded as right-infinite (reduced) words in the alphabet $\mathcal{B} \cup \bar{\mathcal{B}}$. Further, any (finite) word w determines a clopen (closed and open) subset of $\partial\mathbb{F}$, namely the set $\{w\cdots\}$ of right-infinite words having w as an initial subword. These clopen sets form a basis for the topology on $\partial\mathbb{F}$.

There is a unique left action of $\mathbb{F}$ on $\mathbb{F} \cup \partial\mathbb{F}$ which continuously extends the left multiplication of $\mathbb{F}$ on $\mathbb{F}$. It is left multiplication of right-infinite words.

A *marked graph* G is a graph (CW-complex of dimension at most one) whose vertices have valence at least three and whose fundamental group (after choosing a base point) is identified with $\mathbb{F}$. The universal cover $\tilde{G}$ of a marked graph G is also compactified by $\partial\mathbb{F}$ in a natural way.

A special case is the rose $\mathrm{Rose}(\mathcal{B})$, which is the graph with one vertex and $\mathrm{rk}(\mathbb{F})$ edges, and whose fundamental group with the vertex as basepoint is identified with $\mathbb{F}$ in such a way that the edges correspond to the generators in the basis $\mathcal{B}$. The universal cover of $\mathrm{Rose}(\mathcal{B})$ is the Cayley graph $\mathrm{Cayley}(\mathbb{F}, \mathcal{B})$.

Let us also note at this point that we often will be somewhat sloppy with paths in graphs, regarding them sometimes as (locally injective) maps of real intervals into the graph, sometimes as (reduced) sequences of oriented edges.

5.3 Measured Currents

The space $\partial^2\mathbb{F}$ of *(abstract) geodesics* of $\mathbb{F}$ is defined as $(\partial\mathbb{F} \times \partial\mathbb{F} - \Delta)/\mathbb{Z}_2$, where the nontrivial element of $\mathbb{Z}_2$ acts on $\partial\mathbb{F} \times \partial\mathbb{F}$ by interchanging coordinates, and where Δ denotes the diagonal of $\partial\mathbb{F} \times \partial\mathbb{F}$. In other words, $\partial^2\mathbb{F}$ can be regarded as the space of 2-element subsets of $\partial\mathbb{F}$. The left action of $\mathbb{F}$ on $\partial\mathbb{F}$ induces an left action of $\mathbb{F}$ on $\partial^2\mathbb{F}$.

With respect to some marked graph G, elements of $\partial^2\mathbb{F}$ can be regarded as *(geometric) geodesics* in the universal cover $\tilde{G}$. This are unoriented biinfinite reduced paths in $\tilde{G}$.

Let $\mathcal{C} = \mathcal{C}(\mathbb{F})$ denote the collection of compact open subsets of $\partial^2\mathbb{F}$. A *measured* *current* η is an additive, nonnegative function on $\mathcal{C}(\mathbb{F})$ which is invariant under the action of $\mathbb{F}$. By $\mathcal{MC} = \mathcal{MC}(\mathbb{F})$ denote the set of all measured currents. It has a cone structure in the sense that if $\eta_1, \eta_2 \in \mathcal{MC}(\mathbb{F})$ then $a_1\eta_1 + a_2\eta_2 \in \mathcal{MC}(\mathbb{F})$ for $a_1, a_2 \geq 0$. There is also a natural topology on $\mathcal{MC}(\mathbb{F})$, induced from the product topology on the space $\mathbb{R}^{\mathcal{C}(\mathbb{F})}$ of all real functions on $\mathcal{C}(\mathbb{F})$. So, a subbasis for this topology is given by the sets $\{\eta : |\eta(C) - \eta_0(C)| < \varepsilon\}$, where $\eta_0 \in \mathcal{MC}(\mathbb{F})$, $\varepsilon > 0$ and $C \in \mathcal{C}(\mathbb{F})$.

For any nontrivial w, positive powers of w will converge in $\mathbb{F} \cup \partial\mathbb{F}$ to some element w^∞ of $\partial\mathbb{F}$, and negative powers will converge to some element $\bar{w}^\infty \neq w^\infty$ of $\partial\mathbb{F}$. Thus, each nontrivial $w \in \mathbb{F}$ determines a geodesic $\{w^\infty, \bar{w}^\infty\} \in \partial^2\mathbb{F}$. Given a conjugacy class $\alpha \in [\mathbb{F}]$, we call a geodesic $\{w^\infty, \bar{w}^\infty\}$ with $w \in \alpha$ a *lift* of α.

For each conjugacy class $\alpha \in [\mathbb{F}]$ we want to define the measured current η_α *induced* by α. In the case that α is indivisible, we set $\eta_\alpha(C)$ for $C \in \mathcal{C}(\mathbb{F})$ equal to the number of lifts of α which are in C. In the general case, we can write $\alpha = \beta^k$ with $\beta \in [\mathbb{F}]$ indivisible and $k \geq 1$, and we define $\eta_\alpha = k\eta_\beta$.

A conjugacy class α and its inverse $\bar{\alpha}$ induce the same measured current $\eta_\alpha = \eta_{\bar{\alpha}}$. On the other hand, if a conjugacy class α induces the same measured current as some class β, then either $\alpha = \beta$ or $\alpha = \bar{\beta}$. This follows for example from the material in the next section. Thus, there is a natural inclusion of $[\mathbb{F}]$ modulo inverses into $\mathcal{MC}(\mathbb{F})$.

5.4 Coordinates for the Space of Measured Currents

Consider a marked graph G and its universal cover $\tilde{G}$. Every compact subset S of $\tilde{G}$ determines a compact open subset of $\partial^2\mathbb{F}$, namely the set of (geometric) geodesics which contain S in their image. So we can apply any measured current η to this set to get $\eta(S)$.

In particular, if we have given a basis $\mathcal{B}$ and an element $w \in \mathbb{F}$ regarded as word in $\mathcal{B}$, we can look at the path starting at the origin in $\mathrm{Cayley}(\mathbb{F}, \mathcal{B})$ determined by w. (It consists of a single point in the case $w = 1$.) We can apply η to the set of geodesics $C_w \in \mathcal{C}(\mathbb{F})$ containing this path to get what we just denote by $\eta(w)$. See figure 5.1 Note that this notation only makes sense with respect to a basis $\mathcal{B}$. Also note that η is invariant under inversion, i.e., we have $\eta(w) = \eta(\bar{w})$ for all words w.

The measured current η is uniquely determined by its values on $\mathbb{F}$, since every $C \in \mathcal{C}(\mathbb{F})$ is a finite disjoint union of sets $C_w \in \mathcal{C}(\mathbb{F})$ with $w \in \mathbb{F}$.

For a basic example, consider an indivisible conjugacy class $\alpha \in [\mathbb{F}]$. Then $\eta_\alpha(w)$ is the number of occurrences of w in the cyclic words α and $\bar{\alpha}$. This is done in such a way that for example $\eta_{[ab]}(aba) = 1$.

The additivity of η can be restated as the condition

$$\eta(w) = \sum_{\substack{v \geq w \\ |v| = |w|+1}} \eta(v)$$

for any word $w \in \mathbb{F}$. If $w \neq 1$, the sum has $2\,\mathrm{rk}(\mathbb{F}) - 1$ summands. As an example, for $\mathrm{rk}(\mathbb{F}) = 2$ we have $\eta(a) = \eta(aa) + \eta(ab) + \eta(a\bar{b})$. Further, we have $\eta(1) = \eta(a) + \eta(b) + \cdots + \eta(c)$, and $\eta_\alpha(1) = |\alpha|$ for any $\alpha \in [\mathbb{F}]$.

Figure 5.1: The set $S \times T/\mathbb{Z}^2$ determined by the word *baaba*

Another way of restating additivity is to say that

$$\eta(w) = \sum_{\substack{v \text{ has } w \text{ as terminal subword} \\ |v|=|w|+1}} \eta(v)$$

for any word $w \in \mathbb{F}$.

Another consequence of the additivity is monotonicity, in the sense that $\eta(w) \leq \eta(v)$ for $v \leq w$. In particular, $\eta_\alpha(w) \leq |\alpha|$.

The following lemma gives us a more concrete way to think of measured currents.

Lemma 7 *Any nonnegative function η' on $\mathbb{F}$ invariant under inversion (i.e., with $\eta'(w) = \eta'(\bar{w})$ for all words w) and satisfying the condition*

$$\eta'(w) = \sum_{\substack{v \geq w \\ |v|=|w|+1}} \eta'(v)$$

for all $w \in \mathbb{F}$ determines a measured current.

Proof. Given a set $C \in \mathcal{C}$, we can write it as a finite union of disjoint sets $C_{w_1}, \ldots, C_{w_n} \in \mathcal{C}$, with $w_1, \ldots w_n \in \mathbb{F}$. We define $\eta'(C) = \eta'(w_1) + \cdots + \eta'(w_n)$. It follows from the assumption on η' that $\eta'(C)$ does not depend on the choice of $w_1, \ldots, w_n$, and that η' is additive on $\mathcal{C}$. $\square$

Thus, $\mathcal{MC}(\mathbb{F})$ can be regarded as a closed cone in $\mathbb{R}^{\mathbb{F}}$.

Also, we can now give another example for a measured current η, namely the one given by $\eta(1) = 1$ and

$$\eta(w) = \frac{1}{2\,\mathrm{rk}(\mathbb{F})(2\,\mathrm{rk}(\mathbb{F}) - 1)^{|w|-1}}$$

for nontrivial words w. This current is not induced by any conjugacy class, since $\eta_{[w]}(w^k) \geq 1$, but $\eta(w^k) \to 0$ for $k \to \infty$, for any nontrivial $w \in \mathbb{F}$.

The following is a simple observation on cyclic words.

Lemma 8 *If no power of $w \in \mathbb{F} - \{1\}$ is in $\alpha \in [\mathbb{F}]$, then $\eta_\alpha(w^k) = 0$ for all k large enough.*

Note that we always take the phrase "for large enough k" to mean "holds for all $k \geq K$, with some integer K."

Lemma 9 *Indivisible conjugacy classes regarded as elements of $\mathbb{R}^{\mathbb{F}}$ are linear independent. Thus, $\mathcal{MC}(\mathbb{F})$ is infinite dimensional.*

Proof. Assume the current $r_1\eta_{\alpha_1} + \cdots + r_n\eta_{\alpha_n}$ with real numbers r_i and distinct indivisible $\alpha_i \in [\mathbb{F}]$ is identically zero. Apply this measured current to a word $w_i \in \alpha_i$ to conclude that $r_i = 0$. $\square$

For a concrete example, let $\alpha_k = [a^k b]$. Then we have $\eta_{\alpha_k}(a^l) = \max\{0, k-l+1\}$, and so $\alpha_1, \alpha_2, \alpha_3, \ldots$ are linear independent in $\mathbb{R}^{\mathbb{F}}$.

Proposition 10 *Let η be a measured current which takes on integer values for all words of length n. Then there exist $\alpha_1, \ldots, \alpha_k \in [\mathbb{F}]$ such that $\eta(w) = \eta_{\alpha_1}(w) + \cdots + \eta_{\alpha_k}(w)$ for all words w of length at most n.*

Proof. It will be convenient to transform this into a question about a directed graph G, defined as follows. The vertices of G are the words in $\mathbb{F}$ of length $n-1$, and the edges are the words of length n. The edge u goes from the vertex v to the vertex w, if v is the initial subword of u of length $n-1$, and if w is the terminal subword of u of length $n-1$. Label the edge u by the number $\eta(u)$.

This graph G has a label-preserving and orientation-reversing automorphism, which sends vertices and edges to their inverses in $\mathbb{F}$.

The additivity of η implies that for each vertex the sum of the labels of the incoming edges equals the sum of the labels of the outgoing edges. In the context of graphs (or electrical networks), this property is known as *Kirchhoff's condition.*

Any closed directed path in G determines a conjugacy class of $\mathbb{F}$, as follows. Assume $u_1, u_2, \ldots, u_k$ is the sequence of edges in this path, and let x_i be the initial letter of u_i. Then $\alpha = [x_1 x_2 \cdots x_k]$ is the conjugacy class associated to our path. It has the property that $\eta_\alpha(u)$ is equal to the number of times the path crosses over the edge u or over the edge $\bar{u}$.

Thus, our goal is to find a collection of closed directed paths such that the number of times any of these paths crosses over one of the edges u and $\bar{u}$ is equal to the label $\eta(u) = \eta(\bar{u})$. But the question of the existence of such a collection is, up to a minor modification, just the classical Eulerian path problem for directed graphs, where the existence of a solution is equivalent to Kirchhoff's condition (see for example [13]). The classical simple proof will also serve in our situation, noting that the terminal vertex of an edge u can never be the initial vertex of the edge $\bar{u}$.

□

Corollary 11 *Let η be a measured current which takes on integer values on all words. Then $\eta = \eta_{\alpha_1} + \cdots + \eta_{\alpha_n}$ for some $\alpha_1, \ldots, \alpha_n \in [\mathbb{F}]$.*

Proof. Apply Proposition 10 for each n. Note that, since $\eta(1) = \eta_{\alpha_1}(1) + \cdots + \eta_{\alpha_n}(1) = |\alpha_1| + \cdots + |\alpha_n|$, there are only finitely many choices for the collection $\alpha_1, \ldots, \alpha_n$ of conjugacy classes. Thus, one choice will be made infinitely often, giving us the desired collection. $\square$

Note that this corollary is actually independent of the coordinates, since the assumption of this corollary is equivalent to the statement that $\eta(C)$ is an integer on all compact and open subsets C of $\partial^2\mathbb{F}$.

5.5 The Action of Automorphisms on the Space of Measured Currents

Each outer automorphism $\Phi \in \mathrm{Out}(\mathbb{F})$ induces a homeomorphism Φ_* of $\mathcal{MC}(\mathbb{F})$ via $\Phi_*\eta(C) = \eta(\Phi^{-1}C)$ for $\eta \in \mathcal{MC}(\mathbb{F})$ and $C \in \mathcal{C}(\mathbb{F})$. This homeomorphism is well-defined since measured currents are invariant under the action of $\mathbb{F}$ on $\partial^2\mathbb{F}$. So the group $\mathrm{Out}(\mathbb{F})$ acts from the left on $\mathcal{MC}(\mathbb{F})$.

We see that $\Phi_*\eta_\alpha = \eta_{\Phi\alpha}$ for $\alpha \in [\mathbb{F}]$.

For an example, consider the automorphism Φ of $\mathbb{F}_2$ given by $\Phi(a) = a, \Phi(b) = \bar{a}b$. The inverse is given by $\Phi^{-1}(a) = a, \Phi^{-1}(b) = ab$. It is easy to verify that

$$\Phi^{-1}(\{a\cdots\}) = \{aa\cdots\} \cup \{a\bar{b}\cdots\}.$$

By looking at complements this implies

$$\Phi^{-1}(\{\bar{a}\cdots\}\cup\{b\cdots\}\cup\{\bar{b}\cdots\}) = \{ab\cdots\}\cup\{\bar{a}\cdots\}\cup\{b\cdots\}\cup\{\bar{b}\cdots\}.$$

Recall that $\{w\cdots\}$ denotes the set of right-infinite words having w as an initial subword. We will now also use the notation $\{\cdots\bar{v}|w\cdots\} = \{v\cdots\}\times\{w\cdots\}/\mathbb{Z}^2 \subset \partial^2\mathbb{F}$.

So,

$$\begin{aligned}\Phi^{-1}(\{\cdots|a\cdots\}) &= \Phi^{-1}(\{\cdots a|a\cdots\}\cup\{\cdots b|a\cdots\}\cup\{\cdots\bar{b}|a\cdots\}) \\ &= \{\cdots\bar{b}\bar{a}|aa\cdots\}\cup\{\cdots\bar{b}\bar{a}|a\bar{b}\cdots\}\cup\{\cdots a|aa\cdots\}\cup\{\cdots a|a\bar{b}\cdots\} \\ &\quad \cup\{\cdots\bar{b}|aa\cdots\}\cup\{\cdots\bar{b}|a\bar{b}\cdots\}\cup\{\cdots b|aa\cdots\}\cup\{\cdots b|a\bar{b}\cdots\}.\end{aligned}$$

By the definition of $\Phi_*\eta$ we now get

$$\begin{aligned}\Phi_*\eta(a) &= \eta(\bar{b}a)+\eta(\bar{b}\bar{b})+\eta(aaa)+\eta(aa\bar{b}) \\ &\quad +\eta(\bar{b}aa)+\eta(\bar{b}a\bar{b})+\eta(baa)+\eta(ba\bar{b}).\end{aligned}$$

This procedure can easily be generalized to yield the following lemma.

Lemma 12 *For each $\Phi\in\mathrm{Out}(\mathbb{F})$, the induced map Φ_* acts linearly on $\mathcal{MC}(\mathbb{F})\subset\mathbb{R}^{\mathbb{F}}$, i.e., given a nontrivial word w there is a finite set S of nontrivial words such that*

$$\Phi_*\eta(w) = \sum_{v\in S}\eta(v)$$

for all $\eta\in\mathcal{MC}(\mathbb{F})$.

Proof. This follows from Bounded Cancellation (Proposition 46). Specifically, we can choose S to be the set of words of length between $|w| - C(\Phi)$ and $|w| + C(\Phi)$, where $C(\Phi)$ is the cancellation bound for Φ. $\square$

More generally, any monomorphism $\Phi : \mathbb{F}' \to \mathbb{F}$ induces an injective (continuous) map from $\partial\mathbb{F}'$ to $\partial\mathbb{F}$ (see Appendix A), and thus it also induces an injective and proper map from $\partial\mathbb{F}' \times \partial\mathbb{F}' - \Delta'$ to $\partial\mathbb{F} \times \partial\mathbb{F} - \Delta$. Therefore, each monomorphism Φ (or its outer automorphism class) induces an injective and proper map from $\partial^2\mathbb{F}'$ to $\partial^2\mathbb{F}$, which we still denote by Φ. Finally, we get an induced map $\Phi_* : \mathcal{MC}(\mathbb{F}') \to \mathcal{MC}(\mathbb{F})$ similar as in the case of automorphisms. Specifically, we define $\Phi_*\eta(C) = \eta(\Phi^{-1}C)$ for $\eta \in \mathcal{MC}(\mathbb{F}')$ and $C \in \mathcal{C}(\mathbb{F})$.

The following is easy to check.

Lemma 13 *Let $\Phi : \mathbb{F}' \to \mathbb{F}$ be a monomorphism between two finitely generated free groups. Then the map $\Phi_* : \mathcal{MC}(\mathbb{F}') \to \mathcal{MC}(\mathbb{F})$ is injective and proper, and thus an embedding, whose image is closed.*

Recall that an injective and proper map from any space to a compactly generated Hausdorff space is an embedding, and its image is closed. Further, $\mathcal{MC}(\mathbb{F})$ and $\mathbb{P}\mathcal{MC}(\mathbb{F})$ are first countable, and thus compactly generated.

5.6 Projectivized Measured Currents

The space $\mathcal{MC}(\mathbb{F})$ has the structure of a cone and can therefore be projectivized to $\mathbb{P}\mathcal{MC}(\mathbb{F})$. So, a projective class in $\mathbb{P}\mathcal{MC}(\mathbb{F})$ is of the form $\{\lambda\eta : \lambda > 0\}$ for some $\eta \in \mathcal{MC}(\mathbb{F})$ which is not identically zero. Using standard coordinates, we can also regard $\mathbb{P}\mathcal{MC}(\mathbb{F})$ as the space of measured currents η with $\eta(1) = 1$. This embeds $\mathbb{P}\mathcal{MC}(\mathbb{F})$ into $[0,1]^{\mathbb{F}}$ as an infinite dimensional compact convex subset. Such sets are known to be homeomorphic to the Hilbert cube $[0,1]^{\mathbb{N}}$.

Note that the action of $\mathrm{Out}(\mathbb{F})$ on $\mathcal{MC}(\mathbb{F})$ descends to an action on $\mathbb{P}\mathcal{MC}(\mathbb{F})$, since it commutes with scalar multiplication, i.e., since $\Phi_*(\lambda\eta) = \lambda\Phi_*\eta$ holds for all $\Phi \in \mathrm{Out}(\mathbb{F}), \eta \in \mathcal{MC}(\mathbb{F})$ and $\lambda \geq 0$.

Also, the remarks on monomorphisms at the end of the last section apply. In particular, we have the following variation of Lemma 13.

Lemma 14 *Let $\Phi : \mathbb{F}' \to \mathbb{F}$ be a monomorphism between two finitely generated free groups. Then the map $\Phi_* : \mathbb{P}\mathcal{MC}(\mathbb{F}') \to \mathbb{P}\mathcal{MC}(\mathbb{F})$ is an embedding with closed image.*

We will not always distinguish in notation between $\eta \in \mathcal{MC}(\mathbb{F})$ and its image in $\mathbb{P}\mathcal{MC}(\mathbb{F})$, if the context allows this. Note also that the statements $\eta(w) = 0$ and $\eta(w) > 0$ make sense even for projectivized measured currents $\eta \in \mathbb{P}\mathcal{MC}(\mathbb{F})$.

Lemma 15 *Let η be a measured current, and $n \geq 2$. There is a constant $C = C(\mathrm{rk}(\mathbb{F}), n)$ with the property that if $\eta(u_0) \geq C$ for some $u_0 \in \mathbb{F}$ with $|u_0| = n$, then* *there exists an* $\alpha \in [\mathbb{F}]$ *with* $\eta(w) \geq \eta_\alpha(w)$ *for all* $w \in \mathbb{F}$ *with* $|w| \leq n$.

Proof. Again we will use the directed graph G introduced in the proof of Proposition 10.

Assume that the edge u_0 has a label of at least C, i.e., $\eta(u_0) \geq C$. Let v_0 be the initial vertex of u_0, and consider the terminal vertex v_1 of u_0. It has $2\,\mathrm{rk}(\mathbb{F})-1$ outgoing edges. By Kirchhoff's condition, at least one of these has a label of at least $C/(2\,\mathrm{rk}(\mathbb{F})-1)$. Call this edge u_1, and consider its terminal vertex v_2. Choose one of the outgoing edges having a label of at least $C/(2\,\mathrm{rk}(\mathbb{F})-1)^2$. Continue this procedure inductively, until we have an path $u_0, u_1, \ldots, u_k$ with $v_k = v_i$ for some $i < k$. This gives us a closed path $u_i, u_{i+1}, \ldots, u_{k-1}$ which determines some $\alpha \in [\mathbb{F}]$ as in the proof of Proposition 10.

Since there are $2\,\mathrm{rk}(\mathbb{F})(2\,\mathrm{rk}(\mathbb{F})-1)^{n-2}$ vertices in G, we have

$$k \leq 2\,\mathrm{rk}(\mathbb{F})(2\,\mathrm{rk}(\mathbb{F})-1)^{n-2}.$$

If we choose

$$C = 2(2\,\mathrm{rk}(\mathbb{F})-1)^{2\,\mathrm{rk}(\mathbb{F})(2\,\mathrm{rk}(\mathbb{F})-1)^{n-2}},$$

it follows that every edge which our path crosses has a label of at least 2. Since our path crosses each edge only once, we conclude that $\eta(w) \geq \eta_\alpha(w)$ for all edges w, i.e., words of length n, and therefore for all words of length at most n. $\square$

From Lemma 15 we can deduce the basic fact, due to Bestvina, that each measured current can projectively approximated by conjugacy classes.

Proposition 16 (Bestvina [1]) *The projectivized measured currents η_α induced by conjugacy classes $\alpha \in [\mathbb{F}]$ are dense in $\mathbb{P}\mathcal{MC}(\mathbb{F})$.*

Proof. Let $\eta \in \mathcal{MC}$ and let $n \geq 2$ be some integer. Apply Lemma 15 to $r\eta$, with some number $r > 0$. By induction there exist a finite sum $\eta' = \eta_{\alpha_1} + \cdots + \eta_{\alpha_k}$ of measured currents induced by conjugacy classes such that $\eta'(w) \leq r \cdot \eta(w) \leq \eta'(w) + C(\mathrm{rk}(\mathbb{F}), n)$ for all words w of length at most n. In other words, we have $|\eta(w) - \eta'(w)/r| \leq C(\mathrm{rk}(\mathbb{F}), n)/r$ for words w with $|w| \leq n$. Thus, finite sums of conjugacy classes yield a dense set in $\mathbb{P}\mathcal{MC}$.

It remains to show that any sum $\eta' = \eta_{\alpha_1} + \cdots + \eta_{\alpha_k}$ with $\alpha_i \in [\mathbb{F}]$ can be approximated in $\mathbb{P}\mathcal{MC}$ by projectivized measured currents induced by conjugacy classes. To achieve this, choose elements $w_1 \in \alpha_1, \ldots, w_k \in \alpha_k$. Now observe that η' is the limit of $1/m \cdot \eta_{\beta_m}$ with $m \to \infty$, where $\beta_m = [w_1^{ma_1} \cdots w_k^{ma_k}]$. $\square$

5.7 Measured Currents Supported by Some Subgroup

If H is any subgroup of $\mathbb{F}$, then we let $[H]$ denote the set of conjugacy classes in $[\mathbb{F}]$ which meet H. We say that a projectivized measured current η is *supported* by the subgroup H if it is in the closure of the set of the projectivized measured currents induced by conjugacy classes in $[H]$.

So if η is supported by H, then $\eta(w) = 0$ for each w which is not a subword of some word in H. The converse is not always true. Let H be the subgroup of $\mathbb{F}_3$ generated by the words $c^k a^k b^k c^k$ $(k = 1, 2, \ldots)$. It is easy to check that $\eta_\alpha(a) = \eta_\alpha(b)$ for each $\alpha \in [H]$, and thus for each projectivized measured current supported by H. Since $\eta_{[a]}(a) = 1$ and $\eta_{[a]}(b) = 0$, it follows that $\eta_{[a]}$ is not supported by H. But $\eta_{[a]}$ is positive only on the words $a, a^2, a^3, \ldots$ and their inverses, which all are subwords of elements of H.

It is not hard to check that if $\mathbb{F}'$ is a free factor of $\mathbb{F}$, then a projectivized measured current of $\mathbb{F}$ is supported by $\mathbb{F}'$ if and only if it is in the image of the map $\iota_* : \mathbb{P}\mathcal{MC}(\mathbb{F}') \to \mathbb{P}\mathcal{MC}(\mathbb{F})$ induced by the inclusion map $\iota : \mathbb{F}' \to \mathbb{F}$.

5.8 Primitive Conjugacy Classes

An element of $\mathbb{F}$ is called *primitive* if it belongs to some basis, or equivalently, if it is the image of a letter in $\mathcal{B}_0$ under some automorphism of $\mathbb{F}$. A conjugacy class of $\mathbb{F}$ is called *primitive* if one (and therefore each) of its elements is primitive. An element or conjugacy class is primitive if and only if it is indivisible and has width one.

The commutator $ab\bar{a}\bar{b}$ is an example for an indivisible element in $\mathbb{F}_2$ which is not primitive. This follows for example from Theorem 49 in Appendix B.

Now set $M(\mathbb{F}) = \mathrm{clos}\{\eta_\alpha : \alpha \in [\mathbb{F}] \text{ is primitive}\} \subset \mathbb{P}\mathcal{MC}(\mathbb{F})$. This is the smallest closed set in $\mathbb{P}\mathcal{MC}(\mathbb{F})$ invariant under the $\mathrm{Out}(\mathbb{F})$-action and containing $\eta_{[a]}$.

Therefore we call $M(\mathbb{F})$ the *minimal set* of $\mathbb{F}$.

For the following theorem we will use some material on Whitehead graphs, which can be found in Appendix B.

Theorem 17 *A projectivized measured current induced by some $\alpha \in [\mathbb{F}]$ is in the minimal set $M(\mathbb{F})$ if and only if* $\mathrm{width}(\alpha) < \mathrm{rk}(\mathbb{F})$, *i.e., if and only if each element of α is contained in a proper free factor of $\mathbb{F}$.*

Proof. First assume that $\mathrm{width}([w]) < \mathrm{rk}(\mathbb{F})$. After possibly changing the basis, we can assume that w is a word involving only the letters $b, \ldots, c$. Now consider the homomorphism Φ of $\mathbb{F}$ mapping a to aw and fixing $b, \ldots, c$. It has an inverse mapping a to $a\bar{w}$ and fixing $b, \ldots, c$, so it is actually an automorphism. Now, since $\Phi^k(a) = aw^k$, it is clear that $\lim_{k\to\infty} \Phi_*^k \eta_{[a]}/k = \eta_{[w]}$ in $\mathcal{MC}$, and thus $\lim_{k\to\infty} \Phi_*^k \eta_{[a]} = \eta_{[w]}$ in $\mathbb{P}\mathcal{MC}$. This shows that $\eta_{[w]} \in M(\mathbb{F})$.

To show the converse, assume $\mathrm{width}([w]) = \mathrm{rk}(\mathbb{F})$. By Lemma 48 and Theorem 49, some Whitehead graph of $[w]$ has no cut vertex and is connected. But every cyclic word $[v]$ having w^2 as a subword ($\eta_{[v]}(w^2) > 0$) has a Whitehead graph which has the Whitehead graph above as a subgraph, i.e., has a connected Whitehead graph without cut vertices. Therefore, by Theorem 49, $\eta_{[v]}(w^2) > 0$ implies $\mathrm{width}(v) = \mathrm{rk}(\mathbb{F})$. In particular, we have $\eta_\alpha(w^2) = 0$ for all primitive α. By the definition of the topology on $\mathbb{P}\mathcal{MC}$, this implies $\eta(w^2) = 0$ for all $\eta \in M(\mathbb{F})$. But $\eta_{[w]}(w^2) > 0$, so $\eta_{[w]} \notin M(\mathbb{F})$. $\square$

Corollary 18 *The minimal set $M(\mathbb{F})$ is a proper subset of $\mathbb{P}\mathcal{MC}(\mathbb{F})$ whenever* $\mathrm{rk}(\mathbb{F}) > 1$.

Proof. We just have to observe that whenever the rank of $\mathbb{F}$ is at least two there exists some $\alpha \in [\mathbb{F}]$ with $\mathrm{width}(\alpha) = \mathrm{rk}(\mathbb{F})$. But the Whitehead graph of $\alpha = [ab \cdots cac \ldots b]$ is a circle (see Figure 5.2), thus the width of α is $\mathrm{rk}(\mathbb{F})$. $\square$

Figure 5.2: The Whitehead graph of $[ab \cdots cac \ldots b]$

Corollary 19 *The minimal set $M(\mathbb{F})$ is infinite dimensional if* $\mathrm{rk}(\mathbb{F}) > 2$.

Proof. Let $\mathbb{F}'$ be any free factor of $\mathbb{F}$ with rank two. Theorem 17 shows that projectivized measured currents induced by conjugacy classes in $[\mathbb{F}']$ are in the minimal set $M(\mathbb{F})$. By Lemma 14 and an earlier remark it follows that the image of the embedding $\iota_* : \mathbb{P}\mathcal{MC}(\mathbb{F}') \to \mathbb{P}\mathcal{MC}(\mathbb{F})$ is contained in $M(\mathbb{F})$, where $\iota : \mathbb{F}' \to \mathbb{F}$ is the inclusion. But $\mathbb{P}\mathcal{MC}(\mathbb{F}')$ is infinite dimensional by Lemma 9. $\square$

The situation is quite different for $\mathrm{rk}(\mathbb{F}) = 2$. Identifying $\mathbb{F}_2$ with the fundamental group of a punctured torus T, it is known that each automorphism of $\mathbb{F}_2$ is induced by a homeomorphism of T. Primitive conjugacy classes correspond to isotopy classes of embedded curves. Using this correspondence and material from [11] on measured foliations, it is not hard to see the following.

Proposition 20 *The minimal set $M(\mathbb{F}_2)$ is homeomorphic to the circle S^1.*

In particular, it turns out that $\eta \in M(\mathbb{F}_2)$ is completely determined by the ratios of the three values $\eta(a), \eta(b)$ and $\eta(ab)$.

A necessary condition for any projectivized measured current η uses its Whitehead graph $\mathrm{Wh}(\eta, \mathcal{B})$, defined similarly as for conjugacy classes. See Appendix B.

Proposition 21 *The Whitehead graph with respect to any basis of a projectivized measured current in $M(\mathbb{F})$ has a cut vertex or is disconnected.*

Proof. Let $A(\mathcal{B})$ denote the set of all $\eta \in \mathbb{P}\mathcal{MC}$ such that $\mathrm{Wh}(\eta, \mathcal{B})$ has a cut vertex or is disconnected. Note that a subgraph (with the same vertex set) of a graph which has a cut vertex or is disconnected has itself a cut vertex or is disconnected. Also, there is only a finite number of possible Whitehead graphs. Thus, $A(\mathcal{B})$ is closed. Therefore,

$$A = \bigcap_{\mathcal{B} \text{ a basis}} A(\mathcal{B})$$

is closed. But primitive conjugacy classes induce projectivized measured which are in A. Thus $M(\mathbb{F}) \subset A$, and we are done. $\square$

As an illustration, we show that if $\mathrm{rk}(\mathbb{F}) > 2$, then the projectivization of $\eta_{[a]} + \eta_{[b]} + \cdots + \eta_{[c]} \in \mathcal{MC}(\mathbb{F})$ is in $M(\mathbb{F})$. For simplicity of notation assume $\mathrm{rk}(\mathbb{F}) = 3$.

Consider three automorphisms Φ_1, Φ_2 and Φ_3 which satisfy $\Phi_1(a) = ca, \Phi_2(a) = ab$ and $\Phi_3(b) = ab\bar{a}$, and which fix all other generators. Then $\Phi_3^n \Phi_2^{2n} \Phi_1^{2n}(a) = \Phi_3^n(c^{2n}ab^{2n}) = c^{2n}a^{n+1}b^{2n}\bar{a}^n$, thus $[c^{2n}a^{n+1}b^{2n}\bar{a}^n]$ is primitive for all n. But clearly $\eta_{[c^{2n}a^{n+1}b^{2n}\bar{a}^n]}/(2n)$ converges to $\eta_{[a]} + \eta_{[b]} + \cdots + \eta_{[c]}$, which means that the projectivization of this measured current is in $M(\mathbb{F})$.

CHAPTER 6

Measured Currents and Outer Space

6.1 The Culler-Vogtmann Outer Space

An $\mathbb{R}$-*tree* is an arcwise connected metric space such that every embedded arc is isometric to an interval of the real line $\mathbb{R}$. An $\mathbb{R}$-tree is *simplicial* if it is the underlying space of some simplicial complex. An $\mathbb{F}$-*tree* is an $\mathbb{R}$-tree equipped with a left action of $\mathbb{F}$. It is *free* if only the trivial element of $\mathbb{F}$ acts with a fixed point, and it is *minimal* if some orbit is dense. Two $\mathbb{F}$-trees are *projectively equivalent* if there is an equivariant homothety (isometry up to scale) between the $\mathbb{R}$-trees.

Given an $\mathbb{F}$-tree T, define its *length function* ℓ_T, a real-valued function on $[\mathbb{F}]$, by $\ell_T([w]) = \min_{x \in T} d_T(x, w \cdot x)$ for $[w] \in [\mathbb{F}]$, where d_T is the metric on T.

The Culler-Vogtmann outer space $\mathrm{CV} = \mathrm{CV}(\mathbb{F})$ consists of all projective classes of minimal free simplicial $\mathbb{F}$-trees. These classes have distinct and strictly positive length functions, so we can regard the space $\mathrm{CV}(\mathbb{F})$ as subspace of the projectivized space of length functions, equipped with the topology induced from the product topology of $\mathbb{R}^{[\mathbb{F}]}$. This enables us to compactify $\mathrm{CV}(\mathbb{F})$ to $\overline{\mathrm{CV}}(\mathbb{F})$. It can be shown that elements of the boundary $\overline{\mathrm{CV}}(\mathbb{F}) - \mathrm{CV}(\mathbb{F})$ are length functions of $\mathbb{F}$-trees (which must be not free or not simplicial).

The group $\mathrm{Out}(\mathbb{F})$ acts in an natural way from the right on $\mathrm{CV}(\mathbb{F})$. We have $\ell_{x\Phi}(\alpha) = \ell_x(\Phi\alpha)$ for $x \in \mathrm{CV}(\mathbb{F})$ and $\Phi \in \mathrm{Out}(\mathbb{F})$.

There is an another way of defining $\mathrm{CV}(\mathbb{F})$, by considering quotients of $\mathbb{F}$-trees under the $\mathbb{F}$-action. The elements of $\mathrm{CV}(\mathbb{F})$ can be considered as marked metric graphs of volume 1. A *metric* graph has edges with positive lengths assigned. The sum of all the lengths is called the *volume.* The length of an (edge-)path in a metric graph is the sum of the lengths of the edges the path crosses, counted with multiplicities. A *marked graph* is a connected graph whose vertices all have valence at least three and whose fundamental group (after choosing a base point) is identified with $\mathbb{F}$. The group $\mathrm{Out}(\mathbb{F})$ acts on $\mathrm{CV}(\mathbb{F})$ via change of marking.

Note that the Euler characteristic of a marked graph is equal to $1-\mathrm{rk}(\mathbb{F})$. Using the bound on the valence of the vertices, one easily sees that a graph in $\mathrm{CV}(\mathbb{F})$ cannot have more than $3\,\mathrm{rk}(\mathbb{F}) - 3$ edges, and not more than $2\,\mathrm{rk}(\mathbb{F}) - 2$ vertices.

Given any element $x \in \mathrm{CV}(\mathbb{F})$ regarded as $\mathbb{F}$-tree, we have a length function $\ell_x : [\mathbb{F}] \to \mathbb{R}$ as above. If we regard x as marked metric graph, then $\ell_x(\alpha)$ with $\alpha \in [\mathbb{F}]$ is the length of the nontrivial closed reduced path (a *cycle*) representing α via the identification of the fundamental group of the graph with $\mathbb{F}$.

We define a (continuous) map $\sigma : \mathrm{CV}(\mathbb{F}) \to \mathbb{R}$ by $\sigma(x) = \min_{\alpha\in[\mathbb{F}]} \ell_x(\alpha)$. So, σ of the marked metric graph x is equal to the length of the shortest cycle in x. We have $\sigma(x\Phi) = \sigma(x)$ for all $x \in \mathrm{CV}(\mathbb{F})$ and $\Phi \in \mathrm{Out}(\mathbb{F})$.

Lemma 22 *Given a graph $x \in \mathrm{CV}(\mathbb{F})$, there exists a $\Phi \in \mathrm{Out}(\mathbb{F})$ such that*

$$\ell_{x\Phi}(\alpha) \geq \frac{\sigma(x)}{2\,\mathrm{rk}(\mathbb{F}) - 2} \cdot |\alpha|$$

for all $\alpha \in [\mathbb{F}]$.

Proof. Let T be a spanning tree (a maximal subtree) of x of maximal volume. Choose $\Phi \in \mathrm{Out}(\mathbb{F})$ to change the marking in such a way that the edges of x which are not in T correspond to the generators of $\mathbb{F}$ (choose the basepoint in T).

Let l_0 be the length of the shortest of these edges e_0. Then $\ell_{x\Phi}(\alpha) \geq l_0 \cdot |\alpha|$ for all $\alpha \in [\mathbb{F}]$.

Now consider the graph $T \cup e_0$. Since x has at most $2\,\mathrm{rk}(\mathbb{F}) - 2$ vertices, we can choose a cycle in $T \cup e_0$ using at most $2\,\mathrm{rk}(\mathbb{F}) - 2$ edges. Each of these edges has length at most l_0, since otherwise we could replace it in T by e_0, and get a spanning tree of larger volume. Thus $(2\,\mathrm{rk}(\mathbb{F}) - 2)l_0 \geq \sigma(x)$, and therefore $\ell_{x\Phi}(\alpha) \geq \sigma(x)/(2\,\mathrm{rk}(\mathbb{F}) - 2) \cdot |\alpha|$ for all $\alpha \in [\mathbb{F}]$, as desired. $\square$

6.2 Mapping Outer Space Into the Space of Measured Currents

In [7], Bonahon considers an equivariant embedding of the Teichmüller space of a closed orientable surface of genus at least two into the space of geodesic currents of that surface. This is used to give an alternative and simpler construction of Thurston's celebrated compactification of Teichmüller space.

It is not a priori obvious how to translate Bonahon's embedding into the setting of outer space and geodesic currents. The goal of this section is to discuss an equivariant map $\Theta : \mathrm{CV}(\mathbb{F}) \to \mathcal{MC}(\mathbb{F})$.

Let $x \in \mathrm{CV}(\mathbb{F})$. So, x is a marked metric graph of volume 1, with associated length function ℓ_x. We choose a fixed positive number $\theta < 1/(2\,\mathrm{rk}(\mathbb{F}) - 1)^{2\,\mathrm{rk}(\mathbb{F})-2}$, depending only on $\mathrm{rk}(\mathbb{F})$.

Now define Θ by

$$\Theta(x) = \sum_{\alpha \in [\mathbb{F}]} \theta^{\ell_x(\alpha)/\sigma(x)} \eta_\alpha$$

for all $x \in \mathrm{CV}(\mathbb{F})$. So, $\Theta(x)$ is a linear combination of all measured currents induced by conjugacy classes, with high weight on conjugacy classes which are "short" in the marked metric graph x.

Also, let $\mathbb{P}\Theta : \mathrm{CV}(\mathbb{F}) \to \mathbb{P}\mathcal{MC}(\mathbb{F})$ be the map Θ postcomposed with the projectivization map $\mathcal{MC}(\mathbb{F}) \to \mathbb{P}\mathcal{MC}(\mathbb{F})$.

Lemma 23 *The function $\Theta : \mathrm{CV}(\mathbb{F}) \to \mathcal{MC}(\mathbb{F})$ is well-defined and continuous.*

Proof. We have to show that the sum always converges. By Lemma 22 we can choose $\Phi \in \mathrm{Out}(\mathbb{F})$ such that $\ell_{x\Phi}(\alpha) \geq |\alpha| \cdot \sigma(x)/(2\,\mathrm{rk}(\mathbb{F}) - 2)$.

We have

$$\begin{aligned}
\sum_{\alpha\in[\mathbb{F}]} \theta^{\ell_x(\alpha)/\sigma(x)}\eta_\alpha &\le \sum_{\alpha\in[\mathbb{F}]} \theta^{\ell_x(\alpha)/\sigma(x)}|\alpha| \\
&= \sum_{\alpha\in[\mathbb{F}]} \theta^{\ell_x(\Phi\alpha)/\sigma(x)}|\Phi\alpha| \\
&= \sum_{\alpha\in[\mathbb{F}]} \theta^{\ell_{x\Phi}(\alpha)/\sigma(x)}|\Phi\alpha| \\
&\le \sum_{\alpha\in[\mathbb{F}]} \theta^{|\alpha|/(2\,\mathrm{rk}(\mathbb{F})-2)}|\Phi||\alpha| \\
&= |\Phi|\sum_{k=1}^{\infty} q_k k \theta^{k/(2\,\mathrm{rk}(\mathbb{F})-2)},
\end{aligned}$$

where q_k denotes the number of elements of $[\mathbb{F}]$ of length k. But the last sum converges, since by Lemma 50 the number q_k is asymptotic to $(2\,\mathrm{rk}(\mathbb{F})-1)^k/k$.

Continuity of Θ follows from the absolute convergence of the sum, together with the fact that $\ell_x(\alpha)$ and $\sigma(x)$ both depend continuously on x. $\square$

Proposition 24 *Given $x \in \mathrm{CV}(\mathbb{F})$ and $w \in \mathbb{F} - \{1\}$ indivisible and cyclically reduced, we have*

$$\lim_{k\to\infty} \Theta(x)(w^k) = \frac{\theta^{\ell_x([w])/\sigma(x)}}{\left(1-\theta^{\ell_x([w])/\sigma(x)}\right)^2}.$$

Thus, the functions $\Theta : \mathrm{CV}(\mathbb{F}) \to \mathcal{MC}(\mathbb{F})$ and $\mathbb{P}\Theta : \mathrm{CV}(\mathbb{F}) \to \mathbb{P}\mathcal{MC}(\mathbb{F})$ are injective.

Proof. First note that for any $\eta \in \mathcal{MC}$ the sequence $\eta(w^k)$ is monotonically decreasing. In particular, the limit exists. Using Lemma 8 we get

$$\begin{aligned}
\lim_{k\to\infty}\Theta(x)(w^k) &= \sum_{\alpha\in[\mathbb{F}]}\theta^{\ell_x(\alpha)/\sigma(x)}\lim_{k\to\infty}\eta_\alpha(w^k)\\
&= \sum_{i=1}^{\infty}\theta^{\ell_x([w^i])/\sigma(x)}\lim_{k\to\infty}\eta_{[w^i]}(w^k)\\
&= \sum_{i=1}^{\infty} i\theta^{i\ell_x([w])/\sigma(x)}\\
&= \frac{\theta^{\ell_x([w])/\sigma(x)}}{\left(1-\theta^{\ell_x([w])/\sigma(x)}\right)^2}.
\end{aligned}$$

Now note that the function $t\mapsto t/(1-t)^2$ is injective for $0<t<1$. $\square$

Proposition 25 *The function* $\Theta : \mathrm{CV}(\mathbb{F}) \to \mathcal{MC}(\mathbb{F})$ *is equivariant with respect to the action of* $\mathrm{Out}(\mathbb{F})$. *This means that* $\Theta(x\Phi) = \Phi_*^{-1}\Theta(x)$ *for all* $x\in\mathrm{CV}(\mathbb{F})$ *and* $\Phi\in\mathrm{Out}(\mathbb{F})$, *since* $\mathrm{Out}(\mathbb{F})$ *acts from the right on* $\mathrm{CV}(\mathbb{F})$ *and from the left on* $\mathcal{MC}(\mathbb{F})$.

Proof. Choose any $x\in\mathrm{CV}$ and $\Phi\in\mathrm{Out}(\mathbb{F})$. Recall that $\sigma(x\Phi)=\sigma(x)$. We calculate

$$\begin{aligned}
\Theta(x\Phi) &= \sum_{\alpha\in[\mathbb{F}]}\theta^{\ell_{x\Phi}(\alpha)/\sigma(x\Phi)}\eta_\alpha\\
&= \sum_{\alpha\in[\mathbb{F}]}\theta^{\ell_x(\Phi\alpha)/\sigma(x)}\eta_\alpha\\
&= \sum_{\alpha\in[\mathbb{F}]}\theta^{\ell_x(\alpha)/\sigma(x)}\eta_{\Phi^{-1}\alpha}\\
&= \Phi_*^{-1}\sum_{\alpha\in[\mathbb{F}]}\theta^{\ell_x(\alpha)/\sigma(x)}\eta_\alpha\\
&= \Phi_*^{-1}\Theta(x). \quad\square
\end{aligned}$$

6.3 A New Compactification of Outer Space

From Proposition 24 and Proposition 25 we know that $\mathbb{P}\Theta : \mathrm{CV}(\mathbb{F}) \to \mathbb{P}\mathcal{MC}(\mathbb{F})$ is an injective and equivariant map. Thus, we can regard $CV(\mathbb{F})$ as a subspace of $\mathbb{P}\mathcal{MC}(\mathbb{F})$, but with a potentially weaker topology. The closure in $\mathbb{P}\mathcal{MC}(\mathbb{F})$ gives us a compactification of $\mathrm{CV}(\mathbb{F})$.

We are going to show that this compactification is new.

Proposition 26 *The compactification of* $\mathrm{CV}(\mathbb{F})$ *in* $\mathbb{P}\mathcal{MC}(\mathbb{F})$ *is not homeomorphic to the compactification of* $\mathrm{CV}(\mathbb{F})$ *in* $\mathbb{R}^{[\mathbb{F}]}$ *via length function.*

Proof. For positive numbers $r_1, r_2, \ldots, r_{\mathrm{rk}(\mathbb{F})}$ with $r_1 + r_2 + \cdots + r_{\mathrm{rk}(\mathbb{F})} = 1$ let $\mathrm{Rose}(r_1, r_2, \ldots, r_{\mathrm{rk}(\mathbb{F})}) \in \mathrm{CV}(\mathbb{F})$ denote the marked metric graph with one vertex and $\mathrm{rk}(\mathbb{F})$ edges, labeled with the generators $a, b, \ldots, c$ of $\mathbb{F}$, and having lengths $r_1, r_2, \ldots, r_n$ respectively.

If $x = \mathrm{Rose}(r_1, r_2, \ldots, r_{\mathrm{rk}(\mathbb{F})})$, then $\ell_x(\alpha) = r_1\eta_\alpha(a) + r_2\eta_\alpha(b) + \cdots + r_{\mathrm{rk}(\mathbb{F})}\eta_\alpha(c)$, and also $\sigma(x) = \min\{r_1, r_2, \ldots, r_{\mathrm{rk}(\mathbb{F})}\}$.

Let us now assume $\mathrm{rk}(\mathbb{F}) = 3$ for notational simplicity. For $0 < \varepsilon < 1/3$, set $x_\varepsilon = \mathrm{Rose}(2\varepsilon, \varepsilon, 1 - 3\varepsilon)$ and $y_\varepsilon = \mathrm{Rose}(\varepsilon, 2\varepsilon, 1 - 3\varepsilon)$. So, $\sigma(x_\varepsilon) = \varepsilon$.

We get

$$\begin{aligned}
\lim_{\varepsilon\to 0+}\Theta(x_\varepsilon) &= \lim_{\varepsilon\to 0+}\sum_{\alpha\in[\mathbb{F}_3]}\theta^{2\eta_\alpha(a)+\eta_\alpha(b)+(1/\varepsilon-3)\eta_\alpha(c)}\eta_\alpha \\
&= \sum_{\alpha\in[\mathbb{F}_3],\ \eta_\alpha(c)=0}\theta^{2\eta_\alpha(a)+\eta_\alpha(b)}\eta_\alpha \\
&= \sum_{\alpha\in[\mathbb{F}_2]}\theta^{2\eta_\alpha(a)+\eta_\alpha(b)}\iota_*\eta_\alpha \\
&= \sum_{\alpha\in[\mathbb{F}_2]}\theta^{\ell_{\mathrm{Rose}(2/3,1/3)}(\alpha)/\sigma(\mathrm{Rose}(2/3,1/3))}\iota_*\eta_\alpha \\
&= \iota_*\Theta(\mathrm{Rose}(2/3,1/3)),
\end{aligned}$$

where $\iota:\mathbb{F}_2\to\mathbb{F}_3$ is the natural inclusion, sending the generators of $\mathbb{F}_2$ to the first two generators of $\mathbb{F}_3$.

Similarly,

$$\lim_{\varepsilon\to 0+}\Theta(y_\varepsilon) = \iota_*\Theta(\mathrm{Rose}(1/3,2/3)).$$

Now, from Lemma 24 and Lemma 14 it follows that $\Theta(x_\varepsilon)$ and $\Theta(y_\varepsilon)$ converge to distinct elements in $\mathbb{P}\mathcal{MC}(\mathbb{F}_3)$, even though x_ε and y_ε both converge to the same element in $\mathrm{CV}(\mathbb{F}_3)$ compactified via length functions. $\square$

Proposition 27 *The compactification of* $\mathrm{CV}(\mathbb{F})$ *in* $\mathbb{P}\mathcal{MC}(\mathbb{F})$ *contains the minimal set* $M(\mathbb{F})$.

Proof. Since this compactification is $\mathrm{Out}(\mathbb{F})$-invariant, we only need to show that it contains $\eta_{[a]}$.

But a calculation similar as in the proof of Proposition 26 shows that for

$$x_\varepsilon = \mathrm{Rose}\left(\varepsilon, \frac{1-\varepsilon}{\mathrm{rk}(\mathbb{F})-1}, \ldots, \frac{1-\varepsilon}{\mathrm{rk}(\mathbb{F})-1}\right) \in \mathrm{CV}(\mathbb{F})$$

we have

$$\lim_{\varepsilon \to 0+} \Theta(x_\varepsilon) = \eta_{[a]}. \quad \square$$

CHAPTER 7

The Dynamics of Free Group Automorphisms

7.1 The Fixed Set

Given an outer automorphism Φ of $\mathbb{F}$, we denote by $\operatorname{Fix}\Phi_*$ the set of all $\eta \in \mathbb{P}\mathcal{MC}(\mathbb{F})$ with $\Phi_*\eta = \eta$. Since $\mathbb{P}\mathcal{MC}(\mathbb{F})$ is a Hausdorff space, $\operatorname{Fix}\Phi_*$ is always closed, and thus compact.

We write $\operatorname{Fix}\Phi$ for the set of all nontrivial conjugacy classes that are fixed by Φ, but we also use the same notation for the image of this set in $\mathcal{MC}(\mathbb{F})$ as well as in $\mathbb{P}\mathcal{MC}(\mathbb{F})$.

Since $\operatorname{Fix}\Phi_*$ is closed, and since $\Phi\alpha = \alpha$ implies $\Phi_*\eta_\alpha = \eta_\alpha$, it follows that the closure of $\operatorname{Fix}\Phi$ in $\mathbb{P}\mathcal{MC}$ is contained in $\operatorname{Fix}\Phi_*$.

Also note that if Φ_* fixes a projectivized measured current induced by a conjugacy class (i.e., $\Phi_*\eta_\alpha = \lambda\eta_\alpha$ in $\mathcal{MC}(\mathbb{F})$ for some $\lambda > 0$), then it fixes it as a measured current (i.e., $\lambda = 1$). This follows from the fact that automorphisms respect indivisibility of words.

Proposition 28 *For any* $\Phi \in \mathrm{Out}(\mathbb{F})$, *the induced map* Φ_* *on* $\mathbb{P}\mathcal{MC}(\mathbb{F})$ *has a fixed point.*

Proof. We can regard $\mathbb{P}\mathcal{MC}$ as a convex compact subset of the topological vector space $\mathbb{R}^{\mathbb{F}}$. Now the Schauer theorem (see [10]) guarantees the existence of a fixed point. $\square$

Thus, we always have $\mathrm{Fix}\,\Phi_* \neq \emptyset$, even though $\mathrm{Fix}\,\Phi = \emptyset$ can occur.

The following is a simple observation is.

Lemma 29 *The convex hull of the set* $\mathrm{Fix}\,\Phi_* \subset \mathbb{P}\mathcal{MC}(\mathbb{F})$ *is invariant under* Φ_*.

7.2 Irreducible Automorphisms

Let $f : X \to X$ be a homeomorphism of the topological space X. We say that f has *hyperbolic dynamics* on X if there exist two distinct points $x^+, x^- \in X$ such that $f^n(x) \to x^+$ with $n \to \infty$ for all $x \in X - \{x^-\}$, and that $f^n(x) \to x^-$ with $n \to -\infty$ for all $x \in X - \{x^+\}$.

By the continuity of f, both x^+ and x^- are fixed points; and if X satisfies the Hausdorff separation axiom, then there are clearly no other fixed points. We call x^+ the *attracting* fixed point, and x^- the *repelling* fixed point.

We will use the terminology of [5]. An outer automorphism Φ of $\mathbb{F}$ is called *reducible* if $\mathbb{F}$ is the free product of proper free factors $\mathbb{F}_1, \ldots, \mathbb{F}_k$ (i.e., $\mathbb{F} = \mathbb{F}_1 * \cdots * \mathbb{F}_k$) such that Φ permutes the conjugacy classes of $\mathbb{F}_1, \ldots, \mathbb{F}_{k-1}$. Otherwise, Φ is called *irreducible.*

A homotopy equivalence $f : \Gamma \to \Gamma$ of a marked graph G induces an outer automorphism Φ of $\mathbb{F}$ via the marking. If it maps vertices to vertices and is locally injective on each edge, we say that f is a *topological representative* for Φ. A topological representative f determines a *transition matrix,* where the $(i,j)^{\text{th}}$ entry gives the number of times that the path we get by restriction f to the j^{th} edge crosses the i^{th} edge. Recall from Perron-Frobenius theory (see for example [24]) that a nonnegative square matrix A is called *irreducible* if $A + A^2 + \cdots + A^k$ is strictly positive for some $k > 0$.

A *turn* in G is an unordered pair of oriented edges with the same initial vertex. If these two edges are distinct, the turn is called *nondegenerate.* A topological representative $f : G \to G$ maps turns to turns in an obvious way. If a turn is not mapped to a degenerate turn by any positive power of f, it is called *legal,* and otherwise *illegal.* A path in G is *legal,* if it does not contain any illegal turns. If the restrictions of f to the edges of G are all legal paths, we call f a *train track map.*

It was conjectured by Thurston and proven by Bestvina and Handel in [5] that any irreducible outer automorphism of $\mathbb{F}$ has a topological representative which is an irreducible train track map. The proof of the next theorem will make essential use of this fact.

A *pseudo-Anosov* outer automorphism of $\mathbb{F}$ is an outer automorphism whose positive powers are all irreducible.

Theorem 30 (Bestvina [1]) *Let Φ be a pseudo-Anosov outer automorphism of $\mathbb{F}$ with no periodic conjugacy classes. Then Φ_* has hyperbolic dynamics on $\mathbb{P}\mathcal{MC}(\mathbb{F})$.*

We will give the proof following Bestvina [1] closely, starting with a series of lemmas.

Let $f : G \to G$ be an irreducible train track map which topologically represents Φ. Let each edge of G have length 1, and let C be an bounded cancellation constant for $f : G \to G$.

Consider some $\alpha \in [\mathbb{F}]$ as a loop in G. The edges in α which are at a distance of at least C away from any illegal turn in α, where the distance is measured on α, belong to the "good part" of α. The number of "good" edges, divided by the length of α, is called the *goodness* $\gamma(\alpha) \in [0, 1]$ of α.

Lemma 31 *There is some $\eta^+ \in \mathbb{P}\mathcal{MC}(\mathbb{F})$ such that $\Phi_*^n \eta_\alpha$ converges with $n \to \infty$ to η^+, for all $\alpha \in [\mathbb{F}]$.*

Moreover, given any neighborhood U of η^+ and any $\varepsilon > 0$, there is some integer N such that $\Phi_*^N \eta_\alpha \in U$ for all $\alpha \in [\mathbb{F}]$ with $\gamma(\alpha) \geq \varepsilon$.

Proof. Given an edge e_j of G and a power f^N, the number of times that the path determined by f^N restricted to e_j crosses over the edge e_i is given by the $(i,j)^{\text{th}}$ entry of the N^{th} power of the transition matrix.

From this and the fact that the length $|\Phi^n \alpha|$ is unbounded, but the number of illegal turns in α is bounded, the first claim follows easily using the Perron-Frobenius theory on iteration of irreducible nonnegative matrices. $\square$

Applying this lemma to Φ^{-1} instead of Φ, we see that $\Phi_*^{-n}\eta_\alpha$ converges with $n \to \infty$ for all $\alpha \in [\mathbb{F}]$ to the same projectivized measured current η^-.

Note that we only talk about paths which have endpoints at vertices. Also, if v is a path, then we let $[v]$ denote the reduced (locally injective) path homotopic to v relative endpoints.

Lemma 32 *After possibly modifying $f : G \to G$, there exist some integer $M > 0$ such that for any path v but one either $[f^M(v)]$ contains a legal path of length more than $3C$, or it has fewer illegal turns than v.*

Proof. Recall from [5] that a *Nielsen path* v is one whose endpoints are fixed under f and which satisfies $[f(v)] = v$. It is *indivisible* if it has exactly one illegal turn. It is shown in [5] that we can choose the irreducible train track map $f : G \to G$ to have at most one indivisible Nielsen path.

Also note that in our case an indivisible Nielsen path can not form a loop, since this would give a conjugacy class fixed under Φ.

We use the fact that there are only finitely many paths of length at most $3C$. If the lemma would be violated, some definite power of f would have a Nielsen path composed of two indivisible Nielsen paths, a contradiction. $\square$

Lemma 33 *There exist some integer $M > 0$ and some $\delta > 0$ such that for each $\alpha \in [\mathbb{F}]$ either $\gamma(\alpha) \geq \delta$ or*

$$\frac{\textit{number of illegal turns of } \Phi^M \alpha}{\textit{number of illegal turns of } \alpha} \leq 1 - \delta.$$

Proof. Let C_M be the cancellation bound for f^M, where M is as in the previous lemma. Pick some $\alpha \in [\mathbb{F}]$. After canceling, the loop $f^M(\alpha)$ becomes the loop β, say. Choose points on β such that the distance on β of two adjacent points is in $[C_M, 2C_M]$, and look at their preimages on α. This splits α into a couple of paths. From the previous lemma it follows that each path either looses an illegal turn under f^M, or develops a legal path of length at least $3C$, or is an indivisible Nielsen path. No two adjacent paths can be indivisible Nielsen paths, so at least half of all paths are not indivisible Nielsen paths.

Thus, either at least a quarter of all paths lose an illegal turn, or at least a quarter of all paths develop a legal segment of length at least $3C$, each contribute at least C good edges.

In the first case, the number of illegal turns is reduced by a definite proportion, since the lengths of the paths have a definite bound. In the second case, a definite proportion of β is good. $\square$

Lemma 34 *If $\ell_G(\Phi^n\alpha) \geq 3C \cdot \ell_G(\alpha)$, then $\gamma(\Phi^n\alpha) \geq 1/3$.*

Proof. We have

$$\begin{aligned}
\gamma(\Phi^n\alpha) &= \frac{\text{number of good edges in } \Phi^n\alpha}{\ell_G(\Phi^n\alpha)} \\
&\geq \frac{\ell_G(\Phi^n\alpha) - 2C \cdot \text{number of illegal turns in } \Phi^n\alpha}{\ell_G(\Phi^n\alpha)} \\
&\geq \frac{\ell_G(\Phi^n\alpha) - 2C \cdot \text{number of illegal turns in } \alpha}{\ell_G(\Phi^n\alpha)} \\
&\geq \frac{\ell_G(\Phi^n\alpha) - 2C \cdot \ell_G(\alpha)}{\ell_G(\Phi^n\alpha)} \\
&\geq \frac{\ell_G(\Phi^n\alpha) - 2C \cdot \ell_G(\Phi^n\alpha)/3C}{\ell_G(\Phi^n\alpha)} \\
&= \frac{1}{3}. \quad \square
\end{aligned}$$

The following lemma is proved in a similar way.

Lemma 35 *If $\ell_G(\Phi^n\alpha) \geq \ell_G(\alpha)/L$ and*

$$\frac{\textit{number of illegal turns in } \Phi^M\alpha}{\textit{number of illegal turns in } \alpha} \leq \frac{1}{4CL}$$

for some $L > 0$, then $\gamma(\Phi^n\alpha) \geq 1/2$.

Proof. We have

$$\begin{aligned}
\gamma(\Phi^n\alpha) &\geq \frac{\ell_G(\Phi^n\alpha) - 2C \cdot \text{number of illegal turns in } \Phi^n\alpha}{\ell_G(\Phi^n\alpha)} \\
&\geq \frac{\ell_G(\Phi^n\alpha) - \text{number of illegal turns in } \alpha/2L}{\ell_G(\Phi^n\alpha)} \\
&\geq \frac{\ell_G(\Phi^n\alpha) - \ell_G(\alpha)/2L}{\ell_G(\Phi^n\alpha)} \\
&= \frac{1}{2}. \quad \square
\end{aligned}$$

Now we can conclude the proof of Theorem 30.

Let U^+ and U^- be neighborhoods of η^+ resp. η^- in $\mathbb{P}\mathcal{MC}$.

Applying Lemma 34 and Lemma 31 to Φ^{-1}, we see that if the length of α shortens by a definite amount, then a definite iterate of η_α under Φ^{-1} is in U^-. Note that for Φ^{-1} we have a different length function and a different cancellation constant, but this does not pose a problem for us, since the two length functions are related by a bi-Lipschitz condition.

By Lemma 35 and Lemma 33 we see that a definite iterate of α is either in U^+ or the number of illegal turns decreases by a definite (but arbitrarily small) proportion. Together with Lemma 34 it follows now that either a definite iterate of α under Φ is in U^+, or a definite iterate of α under Φ^{-1} is in U^-. This finishes the proof of Theorem 30, since projective measured currents coming from conjugacy classes are dense in $\mathbb{P}\mathcal{MC}$. $\square$

If we restrict ourselves to the minimal set, we can use a similar proof for all pseudo-Anasov outer automorphism, not just the ones which do not fix any conjugacy class.

Theorem 36 *Let Φ be a pseudo-Anosov outer automorphism of $\mathbb{F}$. Then Φ_* has hyperbolic dynamics on the minimal set $M(\mathbb{F})$.*

7.3 Dehn Twists

An outer automorphism Φ is called a *Dehn twist* if it has polynomial growth, in the sense that for each $\alpha \in [\mathbb{F}]$ the expression $|\Phi_*^k \alpha|$ is bounded above by some polynomial in k. We want to start an investigation of the dynamics of the action of a Dehn twist on $\mathbb{P}\mathcal{MC}$.

The homomorphism $\mathbb{F} \to \mathbb{Z}^{\mathrm{rk}\mathbb{F}} \to (\mathbb{Z}/3\mathbb{Z})^{\mathrm{rk}\mathbb{F}}$, constructed by abelianizing and then reducing modulo 3, induces a natural homomorphism from $\mathrm{Aut}(\mathbb{F})$ to $\mathrm{GL}(\mathrm{rk}(\mathbb{F}), \mathbb{Z}/3\mathbb{Z})$. Since inner automorphisms are in the kernel of this homomorphism, it descends to a homomorphism $\mathrm{Out}(\mathbb{F}) \to \mathrm{GL}(\mathrm{rk}(\mathbb{F}), \mathbb{Z}/3\mathbb{Z})$. Recall that an element A of $\mathrm{GL}(n, \mathfrak{K})$ is called unipotent if $(A - \mathrm{Id_n})^n = 0$, where $\mathrm{Id_n}$ denotes the identity matrix of dimension $n \times n$. For our purposes, we will call $\Phi \in \mathrm{Out}(\mathbb{F})$ *unipotent* if its image in $\mathrm{GL}(\mathrm{rk}(\mathbb{F}), \mathbb{Z}/3\mathbb{Z})$ is unipotent.

Since $\mathrm{GL}(\mathrm{rk}(\mathbb{F}), \mathbb{Z}/3\mathbb{Z})$ is a finite group, some power of each $\Phi \in \mathrm{Out}(\mathbb{F})$ is unipotent. This gives some justification to concentrate a study of the dynamics of

Dehn twists to the unipotent case.

Our goal is the following theorem.

Theorem 37 *Let* $\Phi \in \mathrm{Out}(\mathbb{F})$ *be a unipotent Dehn twist. Then the closure of* $\mathrm{Fix}\,\Phi$ *in* $\mathbb{P}\mathcal{MC}$ *is equal to* $\mathrm{Fix}\,\Phi_*$.

Our main tool for the proof will be the following train track representation theorem of M. Bestvina, M. Feighn and M. Handel [4].

Theorem 38 *Let* Φ *be a unipotent Dehn twist. Then there exists some marked graph* Γ *and a homotopy equivalence* $f : \Gamma \to \Gamma$ *representing* Φ, *with the following properties.*

(i) The graph Γ *is filtered by* f*-invariant subgraphs* $\Gamma^0 \subset \Gamma^1 \subset \cdots \Gamma^k = \Gamma$, *where* Γ^i *is obtained from* Γ^{i-1} *by adding an (oriented) edge* e_i.

(ii) The map f *restricted to* Γ^0 *is the identity.*

(iii) For each i, *there is some nontrivial closed reduced path* u_i *in* Γ^{i-1} *such that* $f(e_i) = e_i * u_i$. *Further,* $f^n(e_i) = e_i * u_i * f(u_i) * \cdots * f^{n-1}(u_i)$. *Similar as for words in* $\mathbb{F}$, *we here use the symbol "*" to indicate that there is no canceling. If* $[f(u_i)] = u_i$, *then we call* e_i linear, *otherwise we call it* nonlinear.

(iv) If v *is a reduced path in* Γ *which contains the nonlinear edge* e_i, *then* $f^n(v)$ *contains every initial subpath of the ray* $u_i * f(u_i) * f^2(u_i) * \cdots$ *for all sufficiently large* n.

(v) The edges are partially ordered by setting $e_i < e_j$ if e_i appears in e_j. Let e_i be an edge in the reduced path v which is maximal among all edges in v. Then $v = w * e_i * u$ with paths w and u. If further e_i is nonlinear, then any initial subpath of the ray $e_i * u_i * f(u_i) * f^2(u_i) * \cdots$ is an initial subpath of $f^n(e_i * u)$ for large enough n.

Lemma 39 *Let Φ be a unipotent Dehn twist, let $\eta \in \mathcal{MC}(\mathbb{F})$, and let e_i be an edge maximal among those with $\eta(e_i) > 0$. Then $\Phi_* \eta(e_i) = \eta(e_i)$.*

Proof. By restricting ourselves to a free factor of $\mathbb{F}$, we can assume w.l.o.g. that e_i is maximal among all edges. This equation holds clearly for all measured currents induced by conjugacy classes. The lemma follows then from Proposition 16 and continuity. $\square$

Corollary 40 *If the Dehn twist Φ fixes some measured current η up to scale, it fixes it in the strict sense.*

Proof. Some power of Φ is unipotent. If Φ fixes η up to scale but not in the strict sense, then so does any power of Φ. Thus, we can assume without loss that Φ itself is unipotent.

Choose an edge e_i maximal among those with $\eta(e_i) > 0$. Then $\eta(e_i) = \Phi_* \eta(e_i) = \lambda \eta(e_i)$, and thus $\lambda = 1$. $\square$

If $\mathcal{D}$ is a collection of compact open subsets of $\partial^2\mathbb{F}$, i.e., a subset of $\mathcal{C}(\mathbb{F})$, then we define $\mathcal{MC}_{\mathcal{D}}(\mathbb{F})$ resp. $\mathbb{P}\mathcal{MC}_{\mathcal{D}}(\mathbb{F})$ to be the set of all η in $\mathcal{MC}(\mathbb{F})$ resp. in $\mathbb{P}\mathcal{MC}_{\mathcal{D}}(\mathbb{F})$ which vanish on each element of $\mathcal{D}$. Note that this set is always closed, by the definition of the topology on $\mathcal{MC}(\mathbb{F})$ resp. on $\mathbb{P}\mathcal{MC}(\mathbb{F})$.

Now we consider a unipotent Dehn twist Φ with $f : \Gamma \to \Gamma$ representing Φ as in Theorem 38. For the proof of Theorem 37 we will use a certain collection $\mathcal{D} \subset \mathcal{C}(\mathbb{F})$, which we will describe by reduced paths in Γ. All nonlinear edges are contained in $\mathcal{D}$. Further, for each linear edge e_i, the collection $\mathcal{D}$ contains all reduced paths of the form $e_i * v$, except those which contain an initial subpath $e_i * u_i^n * \bar{e}_j$ or which are contained as initial subpath in $e_i * u_i^n * \bar{e}_j$, for some n and some edge e_j with $u_j = u_i$.

Lemma 41 *A conjugacy class $\alpha \in [\mathbb{F}]$ is fixed by Φ if and only if $\eta_\alpha \in \mathbb{P}\mathcal{MC}_{\mathcal{D}}(\mathbb{F})$.*

Proof. First assume that $\operatorname{Fix}\alpha = \alpha$. Take some edge e_i maximal in α. It is clear from Theorem 38 that for α to be fixed it is necessary that this edge is either fixed by f, or else can be continued to a path $e_i * u_i^n * \bar{e}_j$, for some edge e_j with $u_j = u_i$, and that the path we get after removing $e_i * u_i^n * \bar{e}_j$ from α is fixed (up to homotopy relative end points). Now we apply recursively the same reasoning to this new path, and to u_i, which is also fixed. This induction indeed shows that $\eta_\alpha \in \mathbb{P}\mathcal{MC}_{\mathcal{D}}$.

Now assume $\eta_\alpha \in \mathbb{P}\mathcal{MC}_{\mathcal{D}}$, and pick a maximal edge of α. If e_i is not constant, it is linear. In that case, choose a large subpath of α starting with e_i.

Since $\eta_\alpha \in \mathbb{P}\mathcal{MC}_\mathcal{D}$, this path (if it is long enough) containes an initial subpath of the form $e_i * u_i^n * \bar{e}_j$, with $u_j = u_i$, which is fixed (up to homotopy relative endpoints) under f. Remove this path from α, and continue by induction. This shows that α is composed by paths fixed under f, and so $\alpha \in \operatorname{Fix}\Phi$. $\square$

Proposition 42 *The closure of* $\operatorname{Fix}\Phi$ *in* $\mathbb{P}\mathcal{MC}(\mathbb{F})$ *is equal to* $\mathbb{P}\mathcal{MC}_\mathcal{D}(\mathbb{F})$.

Proof. By Lemma 41 and since $\mathbb{P}\mathcal{MC}_\mathcal{D}$ is closed, it is enough to show that any $\eta \in \mathbb{P}\mathcal{MC}_\mathcal{D}$ can be approximated by sums of elements of $\operatorname{Fix}\Phi$. This can be done by a slight modification of the argument in the proofs of Lemma 15 and Proposition 16. We use the directed graph from the proof of Lemma 15, with words replaced by paths in Γ, and we use the same Eulerian path argument. In addition we have to make sure that the resulting conjugacy classes yield projectivized measured currents which are indeed in $\mathbb{P}\mathcal{MC}_\mathcal{D}$.

The fact that $\eta \in \mathbb{P}\mathcal{MC}_\mathcal{D}$ allows us to always make the "right" choice, in the sense that if we start a path with the linear edge e_i, then we can extend it to a path $e_i u_i^n \bar{e}_j$ with $u_i = u_j$. $\square$

Lemma 43 *Let G be any marked graph, let $\eta \in \mathcal{MC}(\mathbb{F})$, and let $e*v$ be an arbitrary reduced path starting with some edge e, such that $\eta(e * v) > 0$. Then there is some reduced path $e * v * w * e'$ with $\eta(e * v * w * e') > 0$, where $e' = e$ or $e' = \bar{e}$.*

Proof. Assume that the lemma is violated. Then we have for each $n > \ell_G(v) + 1$ that $\eta(e * v) = \eta(e * v * w_1) + \cdots + \eta(e * v * w_k)$, for some reduced paths $w_1, \ldots, w_k$ of length $n - \ell_g(v) - 1$ which do not contain e or $\bar{e}$. Assume without loss that v does not contain e.

Since the paths $e * v * w_1, \ldots, e * v * w_k$ do not "overlap," we have for each edge e_l in G that

$$\eta(e_l) \geq \mu_{l1}\eta(e * v * w_1) + \cdots + \mu_{lk}\eta(e * v * w_k),$$

where μ_{li} is the number of times the edge e_l appears in $e*v*w_i$. Summing the right-hand side of the equation over all edges e_l yields $n(\eta(e*v*w_1)+\cdots+\eta(e*v*w_k))$, thus $n\eta(e * v)$. Let m denote the number of edges of G. It follows that there exist an edge e_l such that $\eta(e_l) \geq n\eta(e)/m$. In particular, $\sum_{e_l} \eta(e_l) \geq n\eta(e)/m$. This holds for all n, a contradiction to $\eta(e) > 0$. $\square$

Proposition 44 *The set* $\operatorname{Fix}\Phi_*$ *is contained in* $\mathbb{P}\mathcal{MC}_{\mathcal{D}}(\mathbb{F})$.

Proof. Assume $\eta \in \operatorname{Fix}\Phi_*$ but $\eta \notin \mathbb{P}\mathcal{MC}_{\mathcal{D}}$. By Corollary 40 we can regard η as a measured current fixed under Φ_* but not in $\mathcal{MC}_{\mathcal{D}}$.

Then we have $\eta(w) > 0$ for $w = e_i$ with some nonlinear edge e_i, or $w = e_i * v$, where e_i is linear and $e_i * v$ contains an initial subpath $e_i * u_i^n * \bar{e}_j$ or is contained as initial subpath in $e_i * u_i^n * \bar{e}_j$, for some n and some edge e_j with $u_j = u_i$.

Consider the set S of edges which appear in reduced paths on which η does not vanish and which contain w as a subpath. First consider the special case that S contains an edge e_k maximal in Γ. By Lemma 43 we then have $\eta(e'_k * u * e''_k) > 0$ for some reduced path $e'_k * u * e''_k$ containing w, with both e'_k and e''_k being either e_k or $\bar{e}_k$. By Theorem 38 it follows that $\Phi_*^n \eta(e'_k * [f^n(u)] * e''_k) = \eta(e'_k * u * e''_k)$ for all measured currents induced by conjugacy classes, and thus for all measured currents, and in particular for our η.

Since $\eta \in \mathrm{Fix}\, \Phi_*$, we have $\eta(e'_k * [f^n(u)] * e''_k) = \eta(e'_k * u * e''_k) > 0$ for all n. But u contains w, and thus the length of $[f^n(u)]$ tends to infinity. But this easily yields a contradiction, by a similar argument as in the proof of Lemma 43.

In the more general case that S does not contain an edge maximal in Γ, we choose some edge e_k maximal among the edges in S. We now use essentially the same argument as in the special case, using the fact that η can be approximated by measured currents coming from conjugacy classes which vanish on all edges not in S. $\square$

We now have proved the chain of inclusions

$$\mathrm{Fix}\, \Phi_* \subset \mathbb{P}\mathcal{MC}_{\mathcal{D}}(\mathbb{F}) \subset \mathrm{clos}(\mathrm{Fix}\, \Phi) \subset \mathrm{Fix}\, \Phi_*,$$

from which Theorem 37 follows. $\square$

APPENDIX A

Bounded Cancellation

A fundamental observation due to Cooper [8], Thurston and Grayson is the existence of a bound on the amount of cancellation that can occur when an automorphism of a free groups is applied.

This result can be generalized to injective homomorphisms (monomorphisms). We will use Nielsen's method (see Section I.2 in [19] for background material).

Lemma 45 *For each monomorphism $\Phi : \mathbb{F} \to \mathbb{F}$ there exist a constant $L = L(\Phi)$ such that $|w|/L \leq |\Phi w| \leq L \cdot |w|$ for all $w \in \mathbb{F}$.*

Proof. Choose a Nielsen-reduced basis $\mathcal{B}$ of $\Phi(\mathbb{F})$, and let $\Psi \in \mathrm{Aut}(\mathbb{F})$ be such that $\Phi\Psi(\mathcal{B}_0) = \mathcal{B}$. It follows from basic properties of Nielsen-reduced bases that $|v|_B \leq |v|$ for all $v \in \Phi(\mathbb{F})$. Thus

$$\frac{|w|}{|\Psi^{-1}|} \leq |\Psi^{-1}w| = |\Phi w|_{\Phi\Psi\mathcal{B}_0} = |\Phi w|_{\mathcal{B}} \leq |\Phi w| \leq |\Phi| \cdot |w|.$$

So $L = \max\{|\Phi|, 1/|\Psi^{-1}|\}$ does the job. $\square$

Proposition 46 (Bounded Cancellation) *For each monomorphism* $\Phi : \mathbb{F} \to \mathbb{F}$ *there exist a constant* $C = C(\Phi)$ *such that*

$$|\Phi(vw)| \geq |\Phi(v)| + |\Phi(w)| - C$$

for all $v, w \in \mathbb{F}$ *with* $|vw| = |v| + |w|$.

Proof. Assume the statement does not hold. Then for each C there are $v_1, v_2 \in \mathbb{F}$ with $|v_1 v_2| = |v_1| + |v_2|, \Phi(v_1) = w_1 * u, \Phi(v_2) = u * w_2$ and $|u| > C$. Moreover, choose v_1, v_2 such that their images under Φ are of minimal length. This implies $|w_i| < L$ $(i = 1, 2)$, because otherwise we could drop the last letter of v_i. Further, since $|\Phi(v_i)| \geq C$ we have $|v_i| \geq C/L$. Together we get

$$\frac{|\Phi(v_1 v_2)|}{|v_1 v_2|} = \frac{|w_1 w_2|}{|v_1| + |v_2|} < \frac{2L}{2C/L} = \frac{L^2}{C}.$$

This yields a contradiction to Lemma 45 by choosing $C(\Phi) = L^3(\Phi)$. $\square$

Corollary 47 *Each monomorphism* $\Phi : \mathbb{F} \to \mathbb{F}$ *has a unique extension to a continuous map* $\Phi : \mathbb{F} \cup \partial\mathbb{F} \to \mathbb{F} \cup \partial\mathbb{F}$, *and this extension is injective.*

APPENDIX B

Whitehead Graphs and Decomposition Spaces

In this appendix we define the *Whitehead graph* $\mathrm{Wh}(\alpha, \mathcal{B})$ of $\alpha \in [\mathbb{F}]$ with respect to a basis $\mathcal{B}$, and give some of its properties.

For us, a *graph* is a CW-complex of dimension at most one. The set of vertices of $\mathrm{Wh}(\alpha, \mathcal{B})$ is $\mathcal{B} \cup \bar{\mathcal{B}}$. The number of edges between the two distinct vertices x and $y \in \mathcal{B} \cup \bar{\mathcal{B}}$ is $\eta_\alpha(\bar{x}y) = \eta_\alpha(\bar{y}x)$. Thus, $\mathrm{Wh}(\alpha, \mathcal{B})$ has $\mathrm{rk}(\mathbb{F})$ vertices and $|\alpha|$ edges.

We say $\mathrm{Wh}(\alpha, \mathcal{B})$ has a *cut vertex* if one of its vertices is a cut point in the usual topological sense.

We also think of $\mathrm{Wh}(\alpha, \mathcal{B})$ as having not $\eta_\alpha(\bar{x}y)$ edges between x and y, but rather having a single edge labeled by $\eta_\alpha(\bar{x}y)$. This does not affect the notions of cut vertices and of connectedness, which are relevant for us, and so this is an insubstantial modification. Its advantage is that it is applicable to all measured current. Thus, to be explicit, the Whitehead graph $\mathrm{Wh}(\eta, \mathcal{B})$ of $\eta \in \mathcal{MC}(\mathbb{F})$ has $\mathcal{B} \cup \bar{\mathcal{B}}$ as vertex set, and there is an edge labeled $\eta(\bar{x}y)$ between the two distinct vertices x and y if $\eta(\bar{x}y) > 0$.

A basic technical lemma is the following:

Lemma 48 (Whitehead [26]) *If the Whitehead graph* $\mathrm{Wh}(\alpha, \mathcal{B})$ *has a cut ver-*

tex, then there is a basis $\mathcal{B}'$ such that $|\alpha|_{\mathcal{B}'} < |\alpha|_{\mathcal{B}}$. Consequently, for each $\alpha \in [\mathbb{F}]$ there is a basis $\mathcal{B}$ such that $\mathrm{Wh}(\alpha, \mathcal{B})$ *has no cut vertices.*

A more geometric way to think of Whitehead graphs is as follows. To a basis $\mathcal{B}$ of $\mathbb{F}$ we can assign a graph with one vertex and with $\mathrm{rk}(\mathbb{F})$ edges, marked in such a way that each edge corresponds to a generator. Now we attach a 2-cell to this graph according to α, i.e., the attaching map represents α via the marking and is locally injective. The resulting space Γ has a fundamental group which has a presentation with generating set $\mathcal{B}$ and one relation given by α.

Locally, there are $2\,\mathrm{rk}(\mathbb{F})$ initial edges adjacent to the one vertex of Γ. These initial edges form the vertices of the Whitehead graph. There is an edge between two initial edges for each occasion they are adjacent in the edge-path given by α.

Or, in different words, $\mathrm{Wh}(\alpha, \mathcal{B})$ is the link of the one vertex of Γ.

We also will need the *generalized Whitehead graph* $\mathrm{Wh}_n(\alpha, \mathcal{B})$, where $n \geq 1$ is some integer. The vertices of $\mathrm{Wh}_n(\alpha, \mathcal{B})$ are the elements of $\mathbb{F}$ with $\mathcal{B}$-length n. Again, there are $\eta_\alpha(\bar{v}w)$ edges between the distinct vertices v and w. As a special case we have $\mathrm{Wh}_1(\alpha, \mathcal{B}) = \mathrm{Wh}(\alpha, \mathcal{B})$.

Given a symmetric relation R on a topological space X we let X/R denote the quotient space of X by the equivalence relation generated by R, equipped with the quotient topology.

To each conjugacy class $\alpha \in [\mathbb{F}]$ there corresponds a set $\tilde{\alpha} \subset \partial^2\mathbb{F}$ of lifts. It can be regarded as symmetric relation on $\partial\mathbb{F}$, which we also denote by α. The quotient space $\mathbb{F}/\alpha$ is called the *decomposition space* of α.

The following theorem is due to Bestvina [2].

Theorem 49 *For any $w \in \mathbb{F} - \{1\}$ the following statements are equivalent:*

(a) The element w is contained in a proper free factor of $\mathbb{F}$.

(b) The width of w is strictly less than $\mathrm{rk}(\mathbb{F})$.

(c) The decomposition space $\partial\mathbb{F}/[w]$ is disconnected.

(d) For each basis $\mathcal{B}$ the generalized Whitehead graph $\mathrm{Wh}_n([w], \mathcal{B})$ is disconnected for some $n \geq 1$.

(e) Any Whitehead graph of $[w]$ with no cut vertices is disconnected.

(f) There exists a disconnected Whitehead graph of $[w]$.

Proof. Note that (a) and (b) are equivalent by definition, and that (e) implies (f) by Lemma 48.

Now assume (a). After changing the basis we can assume that w is contained in the free factor generated by $\mathcal{B}_0 - \{a\} = \{b, \ldots, c\}$. (So $\eta_{[w]}(a) = 0$.) The set of right-infinite words not starting with an a is open and closed in $\partial\mathbb{F}$, and it is saturated with respect to the equivalence relation generated by $[w]$. Thus this set descents to an open and closed proper nontrivial subset of $\partial\mathbb{F}/[w]$. This proves (c).

Next assume (c), i.e., $\partial\mathbb{F}/[w]$ decomposes into two nonempty open subsets U and V. Let $\tilde{U}$ and $\tilde{V}$ be the preimages under the canonical projection $\partial\mathbb{F} \to \partial\mathbb{F}/[w]$. By the definition of the topology on $\partial\mathbb{F}$ there is some $n \geq 1$ such that $\tilde{U}$ and $\tilde{V}$ are unions of open sets given by words of $\mathcal{B}$-length n. (Recall that each word in a basis determines a clopen subset of $\partial\mathbb{F}$ in a natural way.) This corresponds to a partition of the vertices of $\mathrm{Wh}_n([w],\mathcal{B})$ into two nonempty sets. There is an edge between two vertices v_1 and v_2 if and only if some element in the subset of $\partial\mathbb{F}$ determined by v_1 is related via $[w]$ with some element in the subset of $\partial\mathbb{F}$ determined by v_2. But this means that there can not be an edge between our two nonempty sets of vertices, thus $\mathrm{Wh}_n([w],\mathcal{B})$ is disconnected. This is statement (d).

To show that (d) implies (e) we assume that $\mathrm{Wh}([w],\mathcal{B})$ is connected and has no cut points and show by induction on n that $\mathrm{Wh}_n([w],\mathcal{B})$ is connected.

So assume that $\mathrm{Wh}_{n-1}(\alpha,\mathcal{B})$ is connected and pick two elements $w_1, w_2 \in \mathbb{F}$ of $\mathcal{B}$-length n. Write $w_i = u_i * v_i$ with $|u_i|_{\mathcal{B}} = n-1$ and $|v_i|_{\mathcal{B}} = 1$ for $i = 1, 2$. Then there is a path from u_1 to u_2 in $\mathrm{Wh}_{n-1}(\alpha,\mathcal{B})$, and thus there must be a path from $u_1 * \tilde{v}_1$ to $u_2 * \tilde{v}_2$ in $\mathrm{Wh}_n(\alpha,\mathcal{B})$ for some $\tilde{v}_i$ with $|\tilde{v}_i|_{\mathcal{B}} = 1$.

There is a path from v_i to $\tilde{v}_i$ in $\mathrm{Wh}_1(\alpha,\mathcal{B})$. Now we use that $\mathrm{Wh}_1(\alpha,\mathcal{B})$ has no cut points, and choose this path to avoid the vertex given by the first letter of u_i. This yields a path from $u_i * v_i$ to $u_i * \tilde{v}_i$ in $\mathrm{Wh}_n(\alpha,\mathcal{B})$. So there is a path from $w_1 = u_1 * v_1$ to $w_2 = u_2 * v_2$ in $\mathrm{Wh}_n(\alpha,\mathcal{B})$. Thus $\mathrm{Wh}_n(\alpha,\mathcal{B})$ is connected, which finishes our induction.

Finally assume (f), and choose a basis such that the Whitehead graph of $[w]$ is disconnected. Consider a marked rose Γ with a 2-cell attached according to $[w]$, as described above. So the link of the vertex of Γ is connected. Thus we can partition the initial edges nontrivially into two groups such that the attaching map never crosses from one group to the other by passing through the vertex. Now replace the vertex by two vertices and an edge connecting them such that each group of initial edges is attached to one of these two new vertices. The attaching map of the 2-cell does not cross the new edge. This operation yields a complex which is homotopy equivalent to Γ. By a further homotopy equivalence we get a rose with a 2-cell attached which never crosses one of the edges. But this means that w is contained in a proper free factor of $\mathbb{F}$. So (f) implies (a). $\square$

APPENDIX C

The Number of Words of a Given Length

Lemma 50 *The number p_k of cyclically reduced words of $\mathbb{F}$ of length k (with respect to a fixed basis) is $(2\,\mathrm{rk}(\mathbb{F})-1)^k+(-1)^k(\mathrm{rk}(\mathbb{F})-1)+\mathrm{rk}(\mathbb{F})$. The number q_k of cyclic words (i.e., conjugacy classes) in $\mathbb{F}$ of length k is asymptotic to $(2\,\mathrm{rk}(\mathbb{F})-1)^k/k$, in the sense that $q_k k/(2\,\mathrm{rk}(\mathbb{F})-1)^k \to 1$ if $k\to\infty$.*

Proof. The total number of (reduced) words of length k is clearly $2\,\mathrm{rk}(\mathbb{F})\cdot(2\,\mathrm{rk}(\mathbb{F})-1)^{k-1}$. A word $w=a_1a_2\cdots a_k$ ($a_i\in\mathcal{B}\cup\bar{\mathcal{B}}$) is cyclically reduced unless $a_1=\bar{a}_k$. But the number of words of length k with $a_1=\bar{a}_k$ is equal to the number of words of length k with $a_1=a_k$ minus $2\,\mathrm{rk}(\mathbb{F})$. The subtraction accounts for the $2\,\mathrm{rk}(\mathbb{F})$ words with all identical letters. On the other hand, the number of words of length k with $a_1=a_k$ is equal to p_{k-1}. Thus, we get the identity $p_k=2\,\mathrm{rk}(\mathbb{F})(2\,\mathrm{rk}(\mathbb{F})-1)^{k-1}-p_{k-1}+2\,\mathrm{rk}(\mathbb{F})$. Now $p_k=(2\,\mathrm{rk}(\mathbb{F})-1)^k+(-1)^k(\mathrm{rk}(\mathbb{F})-1)+\mathrm{rk}(\mathbb{F})$ follows by induction.

Let r_k denote the number of indivisible conjugacy classes of length k. It is easy to see that $\sum_{l|k} l r_l = p_k$, thus

$$r_k=\frac{1}{k}\left((2\,\mathrm{rk}(\mathbb{F})-1)^k+(-1)^k+2-\sum_{l|k,\,l<k} l r_l\right).$$

This implies easily $r_k \sim (2\,\mathrm{rk}(\mathbb{F}) - 1)^k/k$. Since $q_k = \sum_{l|k} r_l$, it now follows that $q_k \sim (2\,\mathrm{rk}(\mathbb{F}) - 1)^k/k$. $\square$

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