Research announcement (preliminary version - comments welcome) July 1994

Prime factorization and conjugacy problem in $Out(F_n)$

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Abstract

We define two basic types of outer automorphisms φ of a free group F_n which we call *prime automorphisms*. The first type, called *Dehn twist*, mimics algebraically a Dehn twist along a simple closed curve on a surface. The second type is modeled on a pseudo-Anosov homeomorphisms on part of the surface, while fixing all complementary subsurfaces pointwise. It is called *partial pseudo-Anosov.*

Precise definitions are given in terms of the canonical action of $Out(F_n)$ on the space Λ of *combinatorial laminations* in F_n : A prime automorphism φ has precisely one (weakly) attractive fixed point on Λ , the *limit lamination* $L(\varphi)$.

Two prime automorphisms φ_1 , φ_2 commute if both fix the corresponding pair of limit laminations. If only φ_2 fixes both limit laminations then φ_2 is said to *semi-commute* with φ_1 .

Theorem I: (1) For every $\varphi \in Out(F_n)$ there is an exponent $t \geq 1$ and a factorization

$$\varphi^t = \varphi_r \varphi_{r-1} \cdots \varphi_1$$

as a product of prime automorphisms, such that φ_j semi-commutes with φ_i for all $j > i$.

(b) The above *prime factorization* of φ^t is unique, up to permutation of commuting neighbors and certain canonical assembly operations for Dehn twists.

Theorem II: For any two automorphisms φ , $\varphi' \in Out(F_n)$ the question of whether or not φ and φ' are conjugate in $Out(F_n)$ is algorithmically solvable.

0. Introduction

The purpose of this research announcement is to present the main ingredients needed for the solution of the conjugacy problem for free group automorphisms. Our solution is based on a new *prime factorization* and a subsequently deduced structural analysis of the automorphisms in question, which, if $\varphi \in Out(F_n)$ is "reasonable", can be performed by hand in a "reasonable" amount of time. Although we explain some of the computational aspects here, our main objective is to explain the various new concepts entering the work. Our approach is heavily based on the fundamental work of Bestvina-Handel [BH] and on some earlier work of the author [L1, L2].

Partial solutions of the conjugacy problem, including in particular the irreducible case, have been obtained earlier by J. Los and by Z. Sela, who in particular obtained structural data strongly related to some of what is presented here (compare also [L2]).

1. Laminations in free groups (from [L1, L2])

Let A be a fixed basis of F_n, and consider the set

$$\Sigma = \{ s = \ldots s_{-1} s_0 s_1 \ldots \mid s_i \in A \cup A^{-1} , s_{i-1} \neq s_i^{-1} \}$$

of biinfinite reduced words, provided with (a) the product topology, (b) the shift operator σ, and (c) inversion ($s^{-1} = \ldots s_1 s_0 s_{-1} \ldots$). A *lamination* L is a non-empty subset of Σ which is closed with respect to (a) - (c).

A lamination L is *minimal* if it does not contain any proper sublamination. There are two types of minimal laminations:

I. L is finite: Then $L = \{ \sigma^t(s^{\pm 1}) \mid t = 0, \ldots, q-1 , s = \ldots w w w \ldots \}$ for some $w = s_1 \ldots s_q \in F_n$.

II. L is infinite: Then L has the following "recursive" property: For any $s, s' \in L$ any finite subword of s is also a subword of s'.

$Out(F_n)$ acts canonically on Σ / σ and hence on the space Λ of all laminations in Σ. A lamination L is a *limit lamination* of $\varphi \in Out(F_n)$, if $\varphi(L) = L$ and L is weakly φ-attractive[1]. We denote by $L(\varphi)$ the union of all limit laminations of φ. (This is a finite union, hence $L(\varphi)$ is again a lamination.)

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[1] This means precisely the following: There is an $s \in \Sigma$ with the following two properties:

(1) For some (and hence any) $\Phi \in Aut(F_n)$ representing φ and every $k \geq 0$ there exists $n \geq 0$ such that every symmetric finite subword $w = s_{-n} s_{-n+1} \ldots s_n$ of $s = \ldots s_{-1} s_0 s_1 \ldots$ satisfies $| \Phi(w) | \geq 2n + k$.

(2) For any $k > 0$ every finite subword of any element of L is also a subword of $s_k s_{k+1} \ldots$.

(1.1) Definition: $\varphi \in Out(F_n)$ is *prime* if $L(\varphi)$ is minimal and if φ is *pure*: Every φ-periodic conjugacy class in F_n is actually fixed by φ. If $L(\varphi)$ is finite, then φ is called *(single) Dehn twist.* Otherwise we call φ *partial pseudo-Anosov.*

It follows from [BH] (see also [S]) that every automorphism φ has an exponent $t \geq 1$ such that φ^t is pure. The smallest such exponent will be denoted by $t(\varphi)$. Using [BH] one can compute $t(\varphi)$, although this is not strictly necessary for deciding conjugacy of automorphisms.

2. Dehn twists

Multiple Dehn twist automorphisms φ have been introduced and studied in [CL2] : They arise from a graph of groups decomposition $\mathcal{G}$ of F_n with cyclic edge groups by "twisting" along the edges. More precisely, φ induces the identity on all vertex and edge groups of $\mathcal{G}$, and this property characterizes multiple Dehn twists. A single Dehn twist automorphisms is given in the special case where $\mathcal{G}$ has only one edge. Thus every multiple Dehn twist is canonically the product of pairwise commuting single Dehn twists.

Single Dehn twists can be described alternatively in terms of a suitable basis $B = \{b_1, ..., b_n\}$ of F_n, where φ is given through $\Phi \in Aut(F_n)$ with either

(i) $\Phi|_{F(b_1, ..., b_m)} = id$ and $\Phi|_{F(b_{m+1}, ..., b_n)}$ = conjugation by some $w \in F(b_1, ..., b_m)$, or

(ii) $\Phi|_{F(b_1, ..., b_{n-1})} = id$ and $\Phi(b_n) = b_n w$ for some $w \in F(b_1, ..., b_{n-1})$.

Here w (or w^{-1}) is called the twistor of φ.

Contrary to the custom in [CL2] in this paper we mean by a Dehn twist without further specification a single Dehn twist.

In terms of the Bestvina-Handel theory [BH] every Dehn twist can be represented by a 2-strata relative train track representative, where the transition matrix of every stratum is the identity matrix. In fact,

(2.1) every such automorphism is a multiple Dehn twist.

3. Partial pseudo-Anosovs

Examples come from surface theory: Every *geometric* automorphism (i.e. induced by a homeomorphism h of a surface M^2 with boundary) is a partial pseudo-Anosov, if, in terms of the Nielsen-Thurston decomposition, precisely one of the factors of h is pseudo-Anosov, and all the other ones are the identity (in particular there are no twists on the curves separating the invariant subsurfaces of M^2).

Every (not necessarily geometric) partial pseudo-Anosov has a relative train track representative as in [BH] with precisely one irreducible exponentially growing stratum, and all other transition matrices equal to the identity. However, this this doesn't quite characterize partial pseudo-Anosovs. We obtain:

(3.1) Every 2-strata relative train track map with identity transition matrix on the lower stratum and a primitive (i.e. all positive powers are irreducible and hence grow exponentially) transition matrix on the upper stratum represents a partial pseudo-Anosov automorphism.

However, relative train track maps can be quite deceptive: For example, though every irreducible automorphism (in the sense of [BH]) is a partial pseudo-Anosov, the converse is not true even if the automorphism is represented by a 1-stratum train track map $f: \tau \to \tau$ with primitive transition matrix. However, using the notion of the *fundamental group* $\pi_1 L$ *of a lamination* (well defined up to conjugation in F_n) and the technique for computing it given in [L1, L2] , one derives a quick combinatorial test which gives:

(3.2) Proposition: The question of whether a given automorphism $\varphi \in Out(F_n)$ is irreducible or not is effectively decidable.

The limit lamination $L(\varphi)$ of a partial pseudo-Anosov is *strongly φ-expanding*: For some (and hence any) $\Phi \in Aut(F_n)$ representing φ and every $k \geq 0$ there exists $n \geq 0$ such that every finite subword w of any biinfinite sequence $s \in L(\varphi)$, with $|w| \geq n$, satisfies $|\Phi(w)| \geq |w| + k$. It turns out that this property is the basis for various of the characterisitc properties of a partial pseudo-Anosov. We define:

(3.3) An outer automorphism φ of F_n is called *generalized partial pseudo-Anosov* if $L(\varphi)$ is strongly φ-expanding and if φ is pure.

4. Semi-commuting automorphisms and factorization

(4.1) Definition: If φ_1 and φ_2 are two automorphisms, and φ_2 fixes every σ-orbit of $L(\varphi_1)$, we say that φ_2 *semi-commutes* with φ_1 . We write

$$\varphi = \varphi_r * \varphi_{r-1} * \cdots * \varphi_1$$

for the product $\varphi = \varphi_r \varphi_{r-1} \cdots \varphi_1$ if φ_j semi-commutes with φ_i for all $j > i$, and call this a *factorization* of φ . If all the φ_i are prime, we use the term *prime factorization.*

This terminology derives from the following observation, which, for the special case of geometric automorphisms, follows directly from the definitions .

(4.2) Proposition: Any two automorphisms commute if each semi-commutes with the other.

This statement and also its converse seem to be true if in the definition of semi-commutation we allow that φ_2 permutes the σ-orbits of $L(\varphi_1)$; but we do not use this here.

5. Prime factorization: Existence

Bestvina-Handel [BH] show that every $\varphi \in Out(F_n)$ can be represented by a relative train track map $f: \tau \to \tau$ which can be constructed in finitely many steps. If we raise φ to a sufficiently high power φ^t and refine the decomposition of τ into strata appropriately, then the transition matrix for every stratum is either primitive or else the identity matrix. After a minor modification of τ (we "blow up" certain vertices to edges in order to separate endpoints of edges from distinct strata) we can decompose f into a product $f = f_r f_{r-1} \dots f_1$, such that each f_i leaves all but one of the original strata pointwise fixed, and, if this stratum is not exponentially growing, leaves all but one edge pointwise fixed. Thus we can apply (2.1) and a slight extension of (3.1) to obtain:

(5.1) Proposition: For every $\varphi \in Out(F_n)$ we can derive algorithmically a prime factorization

$$\varphi^{t(\varphi)} = \varphi_r * \varphi_{r-1} * \dots * \varphi_1 \quad .$$

6. Uniqueness of the prime factorization

The factorization of $\varphi^{t(\varphi)}$ into prime factors as in (5.1) is not unique, but it is not far from it either: There are two canonical operations on the prime factors which "improve" a factorization (more precise information on the nature of these operations is given in the sections 8 and 10):

(6.1) Commuting Dehn twists δ_1 , δ_2 with conjugate twistors give a product $\delta_2 * \delta_1$ which is again a (single) Dehn twist. In this case we can replace in (5.1) $\delta_1 * \delta_2$ by their product twist.

(6.2) A product $\delta * \varphi$ of a Dehn twist δ and a partial pseudo-Anosov φ , where δ does not commute with φ , may give again a partial pseudo Anosov with $L(\delta * \varphi) = L(\varphi)$. Then we can again replace in (5.1) $\delta * \varphi$ by their product.

We obtain:

(6.3) Proposition: The prime factorization of $\varphi^{t(\varphi)}$ given in (5.1) is unique, up to permutation of commuting adjacent factors and the operations (6.1) and (6.2).

An even better uniqueness result can be obtained if we require the number of prime factors occuring in (5.1) to be minimal. For our purposes, however, it is more suitable to compose certain prime factors to larger factors:

(6.4) Structure Theorem: For every $\varphi \in \mathrm{Out}(F_n)$ there is a canonical factorization

$$\varphi^{t(\varphi)} = \gamma * \Delta_s * \ldots * \Delta_1 ,$$

where γ is a generalized partial pseudo-Anosov, and every $\Delta_j = \delta_{j,d} * \ldots * \delta_{j,1}$ is a multiple Dehn twist, such that none of the single Dehn twists $\delta_{j,k}$ commutes with any $\Delta_{j'}$ with $j' < j$. The above factorizations can be derived algorithmically.

7. The strategy for the conjugacy problem

The fact that the factorization in (6.4) is canonical means precisely that for conjugate pure automorphisms φ and $\varphi' = \rho^{-1} \varphi \rho$ these factorizations have the same length, and that corresponding factors of the two factorizations are also conjugate by ρ. Thus the conjugacy problem is now reduced to solving the following four subproblems:

(7.1) Solve the conjugacy problem for multiple Dehn twists (see section 8).

(7.2) Solve the conjugacy problem for (generalized) partial pseudo-Anosovs. This is the heart of the work described here (see sections 9 - 11).

(7.3) Compose the given solution of the factors to a solution of the conjugacy problem for $\varphi^{t(\varphi)}$.

(7.4) Extend the solution of the conjugacy problem from $\varphi^{t(\varphi)}$ to φ (see section 12).

8. The conjugacy problem for multiple Dehn twists (joint work with M. M. Cohen)

In [CL3] it is shown that for every multiple Dehn twist Δ_j one can derive algorithmically a graph of groups $\mathcal{G}$ as in section 2 with the following properties:

(8.1) Every edge group injects onto a maximal cyclic subgroup in both of its adjacent vertex groups.

(8.2) No vertex group has more than two adjacent (germs of) edges with conjugate edge groups, and, if there are two, then the corresponding twistors have opposite signs.

Such a graph of groups $\mathcal{G}$ is called *proper*, and it is shown in [CL3] that $\mathcal{G}$ together with the set of twist exponents is a complete and computable set of conjugacy invariants for Δ_j.

If one has a product $\Delta = \Delta_d * \ldots * \Delta_1$ of semi-commuting multiple Dehn twists, then there are (uniquely determined) proper graph of groups decompositions $\mathcal{G}_j$ for each Δ_j, and we can "compose" them as follows: Every vertex group of $\mathcal{G}_d$ is a finitely generated subgroup of F_n and hence decomposes with repect to $\mathcal{G}_{d-1}$ into a sub-graph-of-groups. Every vertex group of this sub-graph-of-groups decomposes now with respect to $\mathcal{G}_{d-2}$, and so on. This gives a very fine decomposition of F_n into a *proper generalized graph of groups* $\widehat{\mathcal{G}}$ which differs from a classical graph of groups in that the edge groups of the j-th level are not contained in any vertex group of any level lower than j. The data of $\widehat{\mathcal{G}}$ can be derived algorithmically from the $\mathcal{G}_j$, and, together with the set of twist exponents, they completely characterize Δ up to conjugacy.

9. The conjugacy problem for pseudo-Anosov automorphisms

A pseudo-Anosov automorphism φ is a partial pseudo-Anosov with the further requirement that $\pi_1 L(\varphi) = F_n$. Pseudo-Anosov automorphisms φ share with irreducible automorphisms the property that there is an (up to rescaling) uniquely determined $\mathbb{R}$-tree $\mathcal{T}$ such that all free simplicial $\mathbb{R}$-tree actions (i.e. all points of Culler-Vogtmann space) converge under iteration of φ towards $\mathcal{T}$. Hence, rather than solving the conjugacy problem for φ we can solve it for the pair $(\varphi, \mathcal{T})$. This much improves the situation, as, the richer the structure of two given objects, the easier it is to prove that they are not conjugate.

All geometric data of $\mathcal{T}$ such as the number of F_n-orbits of branch points or their indices are conjugacy invarints for $(\varphi, \mathcal{T})$. For practical purposes it is often convenient to first compute those, with the purpose to quickly get data which in many cases suffice to prove non-conjugacy. However, we will concentrate here on data which will eventually suffice to completely characterize the conjugacy class of φ.

Notice first that for each representative $\Phi \in \mathrm{Aut}(F_n)$ of φ there is a homothety $H: \mathcal{T} \to \mathcal{T}$ with stretching factor $\lambda > 1$ which *commutes* with Φ in the following sense: For all $w \in F_n$ one has $H\, w = \Phi(w)\, H$ (as actions on $\mathcal{T}$). We define an *eigen ray* ρ to be an H-invariant embedded ray in $\mathcal{T}$, i.e. a ray which starts at an H-fixed point Q and which is stretched by H onto itself. We consider on ρ the initial segment of translates $w\,\rho$ of ρ, and notice that these "repeat" canonically if one applies H to ρ. Thus we concentrate on a fundamental domain $\rho_0 \subset \rho$ with respect to the H-action on ρ. For every $w\,Q \in \rho_0$ we define an *overlap*

$$ov(w\,Q) = \frac{\mathrm{length}(\rho \cap w\,\rho)}{\mathrm{dist}(Q, w\,Q)}$$

which clearly does not change if one replaces $w\,Q$ by $H(w\,Q)$, and is invariant with respect to rescaling of $\mathcal{T}$. We then show:

(9.1) Lemma: For any $C > 0$ there are only finitely many $w\,Q \in \rho_0$ with $op(w\,Q) \geq C$.

(9.2) Lemma: There exists a $C > 0$ such that the set $W(\rho_0 , C) := \{ w \in F_n \mid op(w\,Q) \geq C \}$ generates $\pi_1 L(\varphi) = F_n$.

We can describe the dependence of $W = W(\rho_0 , C)$ on ρ_0 in the following way: If we "slide" ρ_0 along ρ (towards the end of ρ), then W changes precisely if some $w\,Q$ with $w \in W$ coincides with the endpoint of ρ_0 closer to Q , and hence has to be replaced by $H(w\,Q) = \Phi(w)\,Q$. In particular, starting with a set W which is a generating system, one obtains after card(W)-many such changes the set $\Phi(W)$, which is again a generating system. In the course of this sliding procedure we can easily check whether there is a $W(\rho_0' , C')$ with smaller cardinality than W which also generates F_n . As there are only finitely many eigen rays ρ in $\mathcal{T}$ (see [GJLL]), we obtain:

(9.3) Proposition: There is a finite set $W(\varphi) = \{W_1, ..., W_m\}$ of generating systems W_i of F_n which are canonically determined by $\mathcal{T}$ (and hence by φ), up to conjugation in F_n and up to application of Φ^t $(t \in \mathbb{Z})$.

(9.4) Corollary: The centralizer $Cen(\varphi)$ of φ in $Out(F_n)$ is a finite extension of the cyclic subgroup generated by φ .

The set $W(\varphi)$ in (9.3) is a complete and finite set of characterizing data for φ up to conjugation in $Out(F_n)$: For any second automorphism $\varphi' \in Out(F_n)$ we chose an arbitrary generating system from $W(\varphi')$, and transform it by a sequence of Nielsen operations into a basis of F_n . We apply the same sequence of Nielsen operations to each of the generating systems $W \in W(\varphi)$, and we write some (arbitraritly chosen) representatives Φ , $\Phi' \in Aut(F_n)$ with respect to these bases. If for some W the resulting images of the basis elements agree up to a conjugation, then φ and φ' are conjugate, and we have actually determined the conjugating automorphism. If this is not the case for any of the generating systems $W \in W(\varphi)$, then φ and φ' are not conjugate.

It remains to indicate how we can derive the generating systems $W(\varphi)$ in an algorithmic way: Here we crucially use combinatorial train tracks as introduced in [L1, L2] and the direct construction of the invariant $\mathbb{R}$-tree $\mathcal{T}$ through combinatorial train tracks as given in [L3] (see also [GJLL]). In fact, it turns out that an easier, purely combinatorial proof of (9.3) can be given entirely in terms of combinatorial train tracks, avoiding the invariant $\mathbb{R}$-tree altogether. Combinatorial train tracks have the advantage over train track representatives (from [BH]) that for conjugate automorphisms and corresponding invariant combinatorial train tracks τ, τ' one always finds two semi-conjugacies between τ and τ' which are inverse to each other up to a power of φ . This allows us to compute the above described data (in particular the set $W(\varphi)$) directly from the combinatorial train track.

10. Similarity classes for partial pseudo-Anosovs

The restriction of a partial pseudo-Anosov φ to its characteristic subgroup $\pi_1 L(\varphi)$ is a pseudo-Anosov automorphism, and its conjugacy class is the most important conjugacy invariant for φ . Further characteristic data about φ comes from the observation that every subgroup F_m of F_n of rank $m \geq 2$ with the property that all of its conjugacy classes are fixed by φ determines canonically a similarity class $[\Phi] \subset \mathrm{Aut}(F_n)$ for φ with the property that each $\Phi' \in [\Phi]$ fixes some conjugate of F_m pointwise.

(10.1) Definition: Φ , $\Phi' \in \mathrm{Aut}(F_n)$ are *similar* if Φ and Φ' are conjugate in $\mathrm{Aut}(F_n)$ by an inner automorphism.

Similarity classes for lifts of φ correspond geometrically to (possibly empty) Nielsen classes of fixed points of topological representatives of φ . A nice geometric interpretation is given through the Goldstein-Turner graph $D(\varphi)$, see [CL1] , where each connected component corresponds precisely to one similarity class. As this correspondence is canonical, we identify the set of similarity classes with $\pi_0 D(\varphi)$. In particular $D(\varphi)$ gives the possibility to compute many data about similarity classes (e.g. the rank of the fixed subgoups), as the methods described in [CL1] for positive automorphisms extend directly to automorphisms with a relative train track representative. In [GJLL] it is shown that every orbit of branch points of the $\mathbb{R}$-tree $\mathcal{T}$ from the last section corresponds in a 1 - 1 fashion to a similarity class for φ , and we call those similarity classes *relevant*. A purely algebraic definition is given alternatively by requiring that the index (= $2\,\mathrm{rk}(\mathrm{Fix}(\Phi)) + \#\{$equivalence classes of Φ-attractive fixed points at $\infty\} - 2$, see [GJLL]) of a relevant similarity class $[\Phi]$ be non-negative, and that $\mathrm{Fix}(\Phi)$ is not contained in $\mathrm{Fix}(\Phi')$ for any $[\Phi']$ with strictly larger index. The above Nielsen class of fixed points corresponding to a relevant similarity class is non-empty for any topological representative of φ .

(10.2) The inclusion $\pi_1 L(\varphi) \subset F_n$ induces canonically a map $i_\varphi : \pi_0 D(\varphi|_{\pi_1 L(\varphi)}) \to \pi_0 D(\varphi)$.

(10.3) Proposition: The conjugacy class of a partial pseudo-Anosov φ is determined precisely by the following data, which can be computed effectively:
(1) The conjugacy class of $\varphi|_{\pi_1 L(\varphi)}$ in $Out(\pi_1 L(\varphi))$.
(2) The finite set $\{ D_1 , ..., D_p \}$ of relevant similarity classes of φ , and for each D_j the (finite) preimage under the above map i_φ .
(3) For each similarity class $[\Phi|_{\pi_1 L(\varphi)}]$ in one of the preimages $i_\varphi^{-1}(D_j)$ from (2) the induced inclusion $Fix(\Phi|_{\pi_1 L(\varphi)}) \subset Fix(\Phi)$.

Let us specify how, for two given partial pseudo-Anosovs φ , φ' we apply this proposition: We first concentrate on the restricted automorphisms $\varphi_0 = \varphi|_{\pi_1 L(\varphi)}$ and $\varphi'_0 = \varphi'|_{\pi_1 L(\varphi)}$ which are pseudo-Anosov. Using the material described in the previous section we decide whether these are conjugate, and if so, we obtain finitely many possibilities for the conjugator. For each such conjugator ρ we proceed as follows: The automorphism ρ gives a canonical 1 - 1 correspondence between the relevant similarity classes for φ_0 and φ'_0 , and for any two corresponding classes $[\Phi_0]$ and $[\Phi'_0]$ an (up to conjugation) canonically determined isomorphism $\rho : Fix(\Phi_0) \to Fix(\Phi'_0)$. We pick an arbitrary basis V of $Fix(\Phi_0)$ and consider the corresponding basis $\rho(V)$ for $Fix(\Phi'_0)$. We then pick an arbitrary basis B for $Fix(\Phi)$, where $\Phi \in Aut(F_n)$ is determined by $[\Phi] = i_\varphi([\Phi_0])$, perform Whitehead transformations on B until the total length of V as words in B is minimal, and record all the (finitely many) shortest systems of words for V . We repeat the same procedure for $\rho(V)$ and compare the resulting systems. More precisely, this has to be done simultaneously for all $[\Phi_0]$ in the same preimage $i_\varphi^{-1}([\Phi])$. If the obtained lists are the same for all relevant similarity classes $[\Phi] \in \pi_0 D(\varphi)$ (for some conjugator ρ as above), then φ and φ' are conjugate, and conversely. Indeed, this methods gives directly a finite presentation for the centralizer of φ in $Out(F_n)$.

Before considering the case of generalized partial pseudo-Anosovs, we want to describe briefly the effect of multiplying a partial pseudo-Anosov φ from the left by a Dehn twist δ which semi-commutes but does not commute with φ . This can now be described precisely as one of the following two:
(1) The product $\varphi' = \delta \varphi$ is again partial pseudo-Anosov, with $L(\varphi') = L(\varphi)$, but the index of the relevant similarity classes may have changed, or distinct similarity classes may have been united.
(2) The limit lamination $L(\varphi')$ is no longer minimal: It consists of $L(\varphi)$ together with finitely many more leaves, with the property that $L(\varphi'|_{\pi_1 L(\varphi')}) = L(\varphi)$ and $\varphi'|_{\pi_1 L(\varphi')} = \varphi|_{\pi_1 L(\varphi')}$. Again multiplication by δ has created a new relevant similarity class, or else increased the index of one of the already existing relevant similarity classes. However, in contrast to (1) , this change in the relevant similarity classes of φ involves at least on class $[\Phi]$ where $i_\varphi^{-1}([\Phi])$ does not

contain any relevant similarity class from $\varphi|_{\pi_1 L(\varphi)}$.

11. The conjugacy problem for generalized partial pseudo-Anosovs

The solution of the conjugacy problem for generalized partial pseudo-Anosovs γ follows precisely the scheme from the previous section for partial pseudo-Anosovs, with a certain amount of additional complexity:

(1) In general it will not be true that $L(\gamma|_{\pi_1 L(\gamma)}) = L(\gamma)$. Instead, we obtain a sequence of nested γ-invariant laminations $L_0 \subset L_1 \subset \ldots \subset L(\gamma)$ and their fundamental groups, defined iteatively through $L_{j-1} = L(\gamma|_{\pi_1 L_j})$, starting from $L(\gamma)$ and ending at the *core lamination* L_0 which satisfies $L_0 = L(\gamma|_{\pi_1 L_0})$. The length of this sequence is bounded by the number of primes in any prime factorization of γ , and it turns out that $L(\gamma) \setminus L_0$ is finite.

(2) The automorphism $\gamma_0 = \gamma|_{L_0}$ is in general not a pseudo-Anosov: However its limit lamination is, though not minimal, still strongly expanding (see section 3). Thus minor technical modifications of the methods described in section 9 allow us to perform a similar procedure as in the case of pseudo-Anosovs and yield an analogous set $W(\gamma_0)$ of finitely many generating systems for each of the connected components of $\pi_1 L_0$, which characterizes γ_0 up to conjugacy in $Out(\pi_1 L_0)$. In particular it follows that the statement (9.4) is also true for *generalized pseudo-Anosovs*, i.e. generalized partial pseudo-Anosovs φ with $\pi_1 L(\varphi) = F_n$, and this property seems to characterize this class of automorphisms.

(3) We consider the inclusions $\pi_1 L_j \subset \pi_1 L_{j+1}$ as well as $\pi_1 L(\gamma) \subset F_n$, and the induced maps $i_{\gamma,j}$ on their similarity classes as in (10.2) . Just as in Proposition (10.3) we determine from $W(\gamma_0)$ a complete and computable set of conjugacy invariants for $\gamma_1 = \gamma|_{\pi_1 L_1}$ by analysing the preimages under $i_{\gamma,1}$ of the relevant similarity classes for γ_1 and the inclusions of the corresponding fixed subgroups. We repeat the procedure replacing γ_1 by $\gamma_2 = \gamma|_{\pi_1 L_2}$ and γ_0 by γ_1 , and so on. After finitely many steps we have reached the level of F_n and thus found a complete and computable set of conjugacy invariants for γ .

12. The complete set of characterizing data for φ

Given the canonical factorization of $\varphi^{t(\varphi)}$ as in the structure theorem (6.4) , our final task is to (1) compose the data derived in sections 8 and 11 in order to obtain a complete set of effectively computable conjugacy invariants for $\varphi^{t(\varphi)}$, and (2) to extend this to φ :

(1) Recall the generalized graph of groups $\widehat{\mathcal{G}}$ derived in section 8 which, together with the set of twist exponents, completely characterizes the product of the multiple Dehn twist factors of $\varphi^{t(\varphi)}$. For each of the relevant similarity classes for the generalized partial pseudo-Anosov factor γ of $\varphi^{t(\varphi)}$ we consider the fixed subgroup and observe that this is invariant under the action of $\varphi^{t(\varphi)}$ (notice that the twistors for the Dehn twists Δ_j are fixed by γ since γ semi-commutes with Δ_j). The fixed subgroup is a finitely generated subgroup $F_{m_i} \subset F_n$, and $\widehat{\mathcal{G}}$ gives rise to a finite generalized-graph-of-groups decomposition $\widehat{\mathcal{G}}_i$ for F_{m_i} which, together with the twistors, characterizes $\varphi^{t(\varphi)}|_{F_{m_i}}$. The finite set $W(\varphi)$ of prefered generating systems for $\pi_1 L_0(\gamma)$, the generalized-graph-of-groups decompositions for the fixed subgroups of all relevant similarity classes of φ together with the twist exponents, and the data derived in part (3) of section 11, expressed in terms of the $\widehat{\mathcal{G}}_i$, constitute a finite computable data system $S(\varphi^{t(\varphi)})$ which completely characterizes $\varphi^{t(\varphi)}$ up to conjugacy.

(2) As φ conjugates $\varphi^{t(\varphi)}$ to itself, the complete set of data $S(\varphi^{t(\varphi)})$ has to be invariant under φ. In particular φ induces a permutation of the relevant similarity classes for $\varphi^{t(\varphi)}$, and for their fixed subgroups a system of isomorphisms which has finite order. This allows us to reduce the question of whether φ and φ' are conjugate, under the assumption that for some $t \geq 1$ the powers φ^t and φ'^t are conjugate, to the question of whether two given system of finite order automorphisms on subgroups F_{m_i} of F_n are conjugate, where the conjugator has to preserve certain conjugacy classes or elements in these subgroups.

Finally, this question has an effective answer through work of Gersten, Krstic and Vogtmann: They show that, if such a conjugator exists, it can be found by, starting with an arbitrary basis B of F_{m_i}, applying Whitehead automorphisms to B which either reduce a certain complexity, or, if that complexity is minimal, yields one of finitely many bases of F_{m_i}. These are related to each other through Whitehead automorphisms which do not increase the complexity. Summarizing we obtain:

12.5 Theorem: For any two automorphisms φ, $\varphi' \in Out(F_n)$ the question of whether or not φ and φ' are conjugate in $Out(F_n)$ is algorithmically solvable.

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